Finite strain theory

In continuum mechanics, the **finite strain theory**—also called **large strain theory**, or **large deformation theory**—deals with deformations in which both rotations and strains are arbitrarily large, i.e. invalidates the assumptions inherent in infinitesimal strain theory. In this case, the undeformed and deformed configurations of the continuum are significantly different and a clear distinction has to be made between them. This is commonly the case with elastomers, plastically-deforming materials and other fluids and biological soft tissue.

Displacement

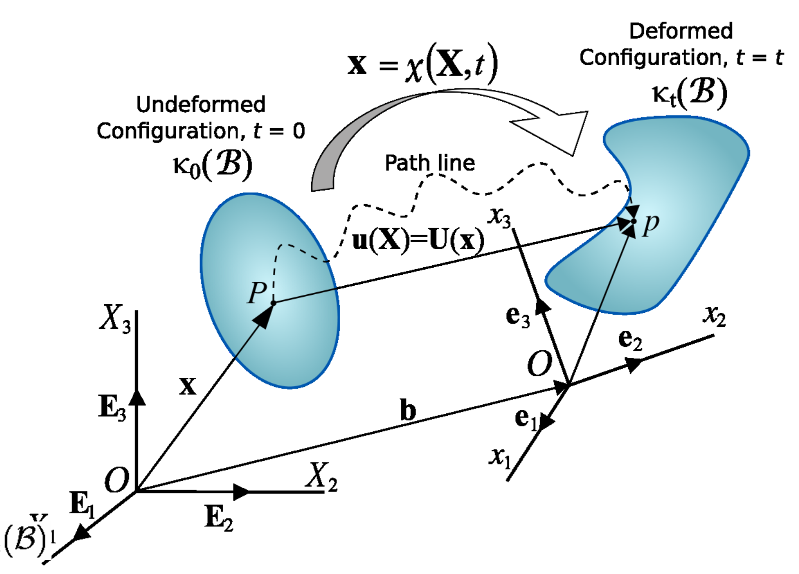

Figure 1. Motion of a continuum body.

A change in the configuration of a continuum body results in a displacement. The displacement of a body has two components: a rigid-body displacement and a deformation. A rigid-body displacement consists of a simultaneous translation and rotation of the body without changing its shape or size. Deformation implies the change in shape and/or size of the body from an initial or undeformed configuration $\kappa_0(\mathcal{B})$ to a current or deformed configuration $\kappa_t(\mathcal{B})$ (Figure 1).

If after a displacement of the continuum there is a relative displacement between particles, a deformation has occurred. On the other hand, if after displacement of the continuum the relative displacement between particles in the current configuration is zero i.e. the distance between particles remains unchanged, then there is no deformation and a rigid-body displacement is said to have occurred.

The vector joining the positions of a particle P in the undeformed configuration and deformed configuration is called the displacement vector $\mathbf{u}(\mathbf{X},t) = u_i\mathbf{e}_i$ in the Lagrangian description, or $\mathbf{U}(\mathbf{x},t) = U_i\mathbf{E}_i$ in the Eulerian description, where $\mathbf{e}_i$ and $\mathbf{E}_i$ are the unit vectors that define the basis of the material (body-frame) and spatial (lab-frame) coordinate systems, respectively.

A *displacement field* is a vector field of all displacement vectors for all particles in the body, which relates the deformed configuration with the undeformed configuration. It is convenient to do the analysis of deformation or motion of a continuum body in terms of the displacement field. In general, the displacement field is expressed in terms of the material coordinates as

$$\mathbf{u}(\mathbf{X},t) = \mathbf{b}(\mathbf{X},t) + \mathbf{x}(\mathbf{X},t) - \mathbf{X} \qquad \text{or} \qquad u_i = \alpha_{iJ}b_J + x_i - \alpha_{iJ}X_J$$

or in terms of the spatial coordinates as

$$\mathbf{U}(\mathbf{x},t) = \mathbf{b}(\mathbf{x},t) + \mathbf{x} - \mathbf{X}(\mathbf{x},t) \qquad \text{or} \qquad U_J = b_J + \alpha_{Ji}x_i - X_J$$

where α_{Ji} are the direction cosines between the material and spatial coordinate systems with unit vectors $\mathbf{E}_J$ and $\mathbf{e}_i$, respectively. Thus

$$\mathbf{E}_J \cdot \mathbf{e}_i = \alpha_{Ji} = \alpha_{iJ}$$

and the relationship between u_i and U_J is then given by

$$u_i = \alpha_{iJ} U_J \qquad \text{or} \qquad U_J = \alpha_{Ji} u_i$$

Knowing that

$$\mathbf{e}_i = \alpha_{iJ} \mathbf{E}_J$$

then

$$\mathbf{u}(\mathbf{X},t) = u_i \mathbf{e}_i = u_i(\alpha_{iJ}\mathbf{E}_J) = U_J\mathbf{E}_J = \mathbf{U}(\mathbf{x},t)$$

It is common to superimpose the coordinate systems for the undeformed and deformed configurations, which results in $\mathbf{b} = 0$, and the direction cosines become Kronecker deltas, i.e.

$$\mathbf{E}_J \cdot \mathbf{e}_i = \delta_{Ji} = \delta_{iJ}$$

Thus, we have

$$\mathbf{u}(\mathbf{X},t) = \mathbf{x}(\mathbf{X},t) - \mathbf{X} \qquad \text{or} \qquad u_i = x_i - \delta_{iJ} X_J$$

or in terms of the spatial coordinates as

$$\mathbf{U}(\mathbf{x},t) = \mathbf{x} - \mathbf{X}(\mathbf{x},t) \qquad \text{or} \qquad U_J = \delta_{Ji} x_i - X_J$$

Displacement gradient tensor

The partial derivative of the displacement vector with respect to the material coordinates yields the *material displacement gradient tensor* $\nabla_{\mathbf{X}}\mathbf{u}$. Thus we have,

$$\begin{aligned}\mathbf{u}(\mathbf{X},t) &= \mathbf{x}(\mathbf{X},t) - \mathbf{X} \\ \nabla_{\mathbf{X}}\mathbf{u} &= \nabla_{\mathbf{X}}\mathbf{x} - \mathbf{I} \\ \nabla_{\mathbf{X}}\mathbf{u} &= \mathbf{F} - \mathbf{I}\end{aligned} \qquad \text{or} \qquad \begin{aligned} u_i &= x_i - \delta_{iJ}X_J = x_i - X_i \\ \frac{\partial u_i}{\partial X_K} &= \frac{\partial x_i}{\partial X_K} - \delta_{iK}\end{aligned}$$

where $\mathbf{F}$ is the *deformation gradient tensor*.

Similarly, the partial derivative of the displacement vector with respect to the spatial coordinates yields the *spatial displacement gradient tensor* $\nabla_{\mathbf{x}}\mathbf{U}$. Thus we have,

$$\begin{aligned}\mathbf{U}(\mathbf{x},t) &= \mathbf{x} - \mathbf{X}(\mathbf{x},t) \\ \nabla_{\mathbf{x}}\mathbf{U} &= \mathbf{I} - \nabla_{\mathbf{x}}\mathbf{X} \\ \nabla_{\mathbf{x}}\mathbf{U} &= \mathbf{I} - \mathbf{F}^{-1}\end{aligned} \qquad \text{or} \qquad \begin{aligned} U_J &= \delta_{Ji}x_i - X_J = x_J - X_J \\ \frac{\partial U_J}{\partial x_k} &= \delta_{Jk} - \frac{\partial X_J}{\partial x_k}\end{aligned}$$

Deformation gradient tensor

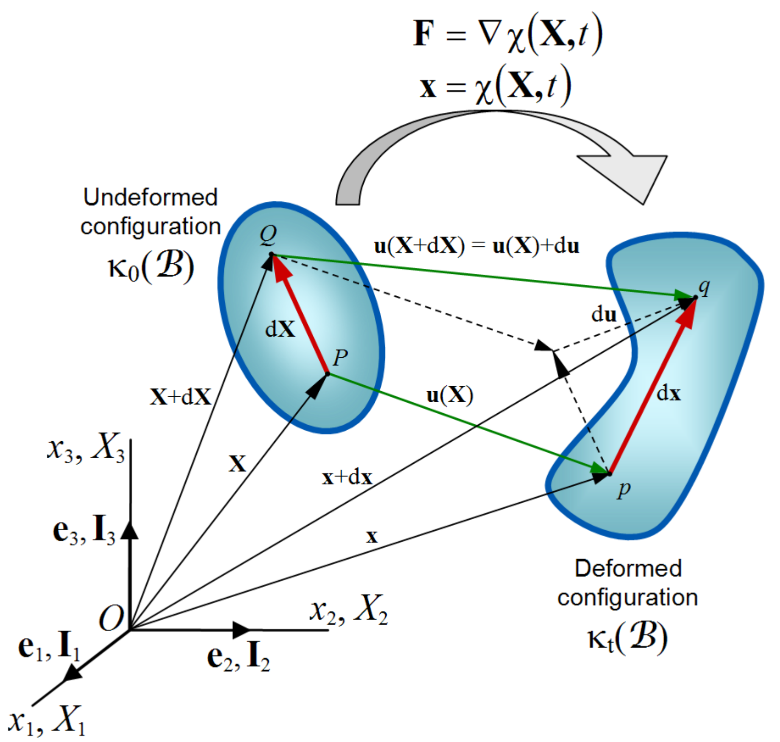

Figure 2. Deformation of a continuum body.

Consider a particle or material point P with position vector $\mathbf{X} = X_I \mathbf{I}_I$ in the undeformed configuration (Figure 2). After a displacement of the body, the new position of the particle indicated by p in the new configuration is given by the vector position $\mathbf{x} = x_i \mathbf{e}_i$. The coordinate systems for the undeformed and deformed configuration can be superimposed for convenience.

Consider now a material point Q neighboring P, with position vector $\mathbf{X} + \Delta\mathbf{X} = (X_I + \Delta X_I)\mathbf{I}_I$. In the deformed configuration this particle has a new position q given by the position vector $\mathbf{x} + \Delta\mathbf{x}$. Assuming that the line segments ΔX and $\Delta\mathbf{x}$ joining the particles P and Q in both the undeformed and deformed configuration, respectively, to be very small, then we can express them as $d\mathbf{X}$ and $d\mathbf{x}$. Thus from Figure 2 we have

$$\begin{aligned}
\mathbf{x} + d\mathbf{x} &= \mathbf{X} + d\mathbf{X} + \mathbf{u}(\mathbf{X} + d\mathbf{X}) \\
d\mathbf{x} &= \mathbf{X} - \mathbf{x} + d\mathbf{X} + \mathbf{u}(\mathbf{X} + d\mathbf{X}) \\
&= d\mathbf{X} + \mathbf{u}(\mathbf{X} + d\mathbf{X}) - \mathbf{u}(\mathbf{X}) \\
&= d\mathbf{X} + d\mathbf{u}
\end{aligned}$$

where $\mathbf{du}$ is the **relative displacement vector**, which represents the relative displacement of Q with respect to P in the deformed configuration.

For an infinitesimal element $d\mathbf{X}$, and assuming continuity on the displacement field, it is possible to use a Taylor series expansion around point P, neglecting higher-order terms, to approximate the components of the relative displacement vector for the neighboring particle Q as

$$\begin{aligned}
\mathbf{u}(\mathbf{X} + d\mathbf{X}) &= \mathbf{u}(\mathbf{X}) + d\mathbf{u} \\
&\approx \mathbf{u}(\mathbf{X}) + \nabla_{\mathbf{X}}\mathbf{u} \cdot d\mathbf{X}
\end{aligned} \quad \text{or} \quad \begin{aligned}
u_i^* &= u_i + du_i \\
&\approx u_i + \frac{\partial u_i}{\partial X_J} dX_J
\end{aligned}$$

Thus, the previous equation $d\mathbf{x} = d\mathbf{X} + d\mathbf{u}$ can be written as

$$\begin{aligned}
d\mathbf{x} &= d\mathbf{X} + d\mathbf{u} \\
&= d\mathbf{X} + \nabla_{\mathbf{X}}\mathbf{u} \cdot d\mathbf{X} \\
&= (\mathbf{I} + \nabla_{\mathbf{X}}\mathbf{u})\, d\mathbf{X} \\
&= \mathbf{F} d\mathbf{X}
\end{aligned}$$

The *material deformation gradient tensor* $\mathbf{F}(\mathbf{X}, t) = F_{jK}\mathbf{e}_j \otimes \mathbf{I}_K$ is a second-order tensor that represents the gradient of the mapping function or functional relation $\chi(\mathbf{X}, t)$, which describes the motion of a continuum. The material deformation gradient tensor characterizes the local deformation at a material point with position vector $\mathbf{X}$, i.e. deformation at neighbouring points, by transforming (linear transformation) a material line element emanating from that point from the reference configuration to the current or deformed configuration, assuming continuity in the mapping function $\chi(\mathbf{X},t)$, i.e. differentiable function of $\mathbf{X}$ and time t, which implies that cracks and voids do not open or close during the deformation. Thus we have,

$$\begin{aligned} d\mathbf{x} &= \frac{\partial \mathbf{x}}{\partial \mathbf{X}}\,d\mathbf{X} \\ &= \nabla\chi(\mathbf{X},t)\,d\mathbf{X} \\ &= \mathbf{F}(\mathbf{X},t)\,d\mathbf{X} \end{aligned} \qquad \text{or} \qquad \begin{aligned} dx_j &= \frac{\partial x_j}{\partial X_K}\,dX_K \\ dx_j &= F_{jK}\,dX_K \end{aligned}$$

The deformation gradient tensor $\mathbf{F}(\mathbf{X},t) = F_{jK}\mathbf{e}_j \otimes \mathbf{I}_K$ is related to both the reference and current configuration, as seen by the unit vectors $\mathbf{e}_j$ and $\mathbf{I}_K$, therefore it is a *two-point tensor*.

Due to the assumption of continuity of $\chi(\mathbf{X},t)$, $\mathbf{F}$ has the inverse $\mathbf{H} = \mathbf{F}^{-1}$, where $\mathbf{H}$ is the *spatial deformation gradient tensor*. Then, by the implicit function theorem (Lubliner), the Jacobian determinant $J(\mathbf{X},t)$ must be nonsingular, i.e. $J(\mathbf{X},t) = \det\mathbf{F}(\mathbf{X},t) \neq 0$

Time-derivative of the deformation gradient

Calculations that involve the time-dependent deformation of a body often require a time derivative of the deformation gradient to be calculated. A geometrically consistent definition of such a derivative requires an excursion into differential geometry[1] but we avoid those issues in this article.

The time derivative of $\mathbf{F}$ is

$$\dot{\mathbf{F}} = \frac{\partial \mathbf{F}}{\partial t} = \frac{\partial}{\partial t}\left[\frac{\partial \mathbf{x}(\mathbf{X},t)}{\partial \mathbf{X}}\right] = \frac{\partial}{\partial \mathbf{X}}\left[\frac{\partial \mathbf{x}(\mathbf{X},t)}{\partial t}\right] = \frac{\partial}{\partial \mathbf{X}}\left[\mathbf{V}(\mathbf{X},t)\right]$$

where $\mathbf{V}$ is the velocity. The derivative on the right hand side represents a **material velocity gradient**. It is common to convert that into a spatial gradient, i.e.,

$$\dot{\mathbf{F}} = \frac{\partial}{\partial \mathbf{X}}\left[\mathbf{V}(\mathbf{X},t)\right] = \frac{\partial}{\partial \mathbf{x}}\left[\mathbf{v}(\mathbf{x},t)\right]\cdot\frac{\partial \mathbf{x}(\mathbf{X},t)}{\partial \mathbf{X}} = \boldsymbol{l}\cdot\mathbf{F}$$

where $\boldsymbol{l}$ is the **spatial velocity gradient**. If the spatial velocity gradient is constant, the above equation can be solved exactly to give

$$\mathbf{F} = e^{\boldsymbol{l}\,t}$$

assuming $\mathbf{F} = \mathbf{1}$ at $t = 0$. There are several methods of computing the exponential above.

Related quantities often used in continuum mechanics are the **rate of deformation tensor** and the **spin tensor** defined, respectively, as:

$$\boldsymbol{d} = \tfrac{1}{2}\left(\boldsymbol{l}+\boldsymbol{l}^T\right)~,~~\boldsymbol{w} = \tfrac{1}{2}\left(\boldsymbol{l}-\boldsymbol{l}^T\right)~.$$

The rate of deformation tensor gives the rate of stretching of line elements while the spin tensor indicates the rate of rotation or vorticity of the motion.

Transformation of a surface and volume element

To transform quantities that are defined with respect to areas in a deformed configuration to those relative to areas in a reference configuration, and vice versa, we use Nanson's relation, expressed as

$$da~\mathbf{n} = J~dA~\mathbf{F}^{-T}\cdot\mathbf{N}$$

where da is an area of a region in the deformed configuration, dA is the same area in the reference configuration, and $\mathbf{n}$ is the outward normal to the area element in the current configuration while $\mathbf{N}$ is the outward normal in the reference configuration, $\mathbf{F}$ is the deformation gradient, and $J = \det\mathbf{F}$.

The corresponding formula for the transformation of the volume element is

$$dv = J~dV$$

Derivation of Nanson's relation

To see how this formula is derived, we start with the oriented area elements in the reference and current configurations:

$$d\mathbf{A} = dA\ \mathbf{N}\ ;\quad d\mathbf{a} = da\ \mathbf{n}$$

The reference and current volumes of an element are

$$dV = d\mathbf{A}^T \cdot d\mathbf{L}\ ;\quad dv = d\mathbf{a}^T \cdot d\mathbf{l}$$

where $d\mathbf{l} = \mathbf{F} \cdot d\mathbf{L}$.

Therefore,

$$d\mathbf{a}^T \cdot d\mathbf{l} = dv = J\ dV = J\ d\mathbf{A}^T \cdot d\mathbf{L}$$

or,

$$d\mathbf{a}^T \cdot \mathbf{F} \cdot d\mathbf{L} = dv = J\ dV = J\ d\mathbf{A}^T \cdot d\mathbf{L}$$

so,

$$d\mathbf{a}^T \cdot \mathbf{F} = J\ d\mathbf{A}^T$$

So we get

$$d\mathbf{a} = J\ \mathbf{F}^{-T} \cdot d\mathbf{A}$$

or,

$$da\ \mathbf{n} = J\ dA\ \mathbf{F}^{-T} \cdot \mathbf{N} \qquad \square$$

Polar decomposition of the deformation gradient tensor

The deformation gradient $\mathbf{F}$, like any second-order tensor, can be decomposed, using the polar decomposition theorem, into a product of two second-order tensors (Truesdell and Noll, 1965): an orthogonal tensor and a positive definite symmetric tensor, i.e.

$$\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{R}$$

where the tensor $\mathbf{R}$ is a proper orthogonal tensor, i.e. $\mathbf{R}^{-1} = \mathbf{R}^T$ and $\det \mathbf{R} = +1$, representing a rotation; the tensor $\mathbf{U}$ is the *right stretch tensor*; and $\mathbf{V}$ the *left stretch tensor*. The terms *right* and *left* means that they are to the right and left of the rotation tensor $\mathbf{R}$, respectively. $\mathbf{U}$ and $\mathbf{V}$ are both positive definite, i.e. $\mathbf{x} \cdot \mathbf{U} \cdot \mathbf{x} \geq 0$ and $\mathbf{x} \cdot \mathbf{V} \cdot \mathbf{x} \geq 0$, and symmetric tensors, i.e. $\mathbf{U} = \mathbf{U}^T$ and $\mathbf{V} = \mathbf{V}^T$, of second order.

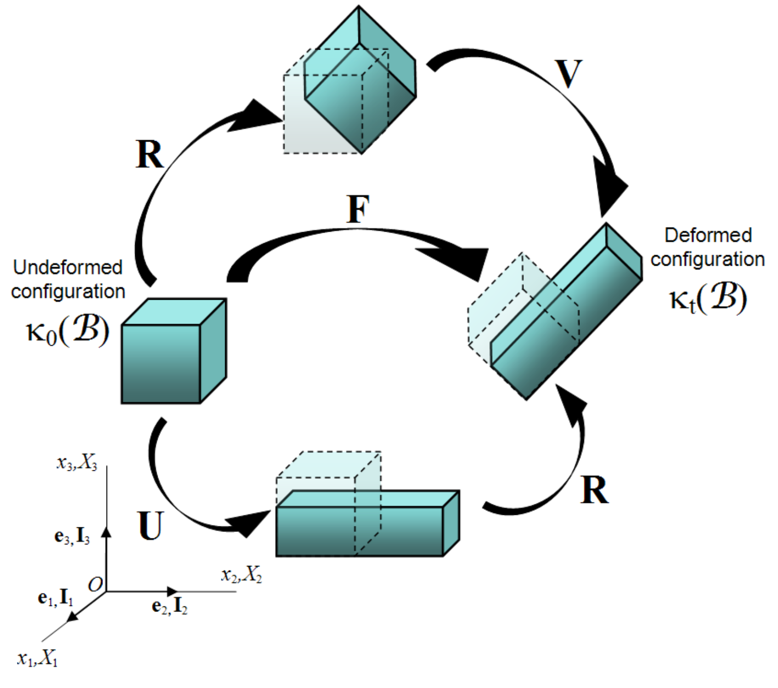

Figure 3. Representation of the polar decomposition of the deformation gradient

This decomposition implies that the deformation of a line element $d\mathbf{X}$ in the undeformed configuration onto $d\mathbf{x}$ in the deformed configuration, i.e. $d\mathbf{x} = \mathbf{F}\,d\mathbf{X}$, may be obtained either by first stretching the element by $\mathbf{U}$, i.e. $d\mathbf{x}' = \mathbf{U}\,d\mathbf{X}$, followed by a rotation $\mathbf{R}$, i.e. $d\mathbf{x} = \mathbf{R}\,d\mathbf{x}'$; or equivalently, by applying a rigid rotation $\mathbf{R}$ first, i.e. $d\mathbf{x}' = \mathbf{R}\,d\mathbf{X}$, followed later by a stretching $\mathbf{V}$, i.e. $d\mathbf{x} = \mathbf{V}\,d\mathbf{x}'$ (See Figure 3).

It can be shown that,

$$\mathbf{V} = \mathbf{R} \cdot \mathbf{U} \cdot \mathbf{R}^T$$

so that $\mathbf{U}$ and $\mathbf{V}$ have the same eigenvalues or principal stretches, but different eigenvectors or *principal directions* $\mathbf{N}_i$ and $\mathbf{n}_i$, respectively. The principal directions are related by

$$\mathbf{n}_i = \mathbf{R}\mathbf{N}_i.$$

This polar decomposition is unique as $\mathbf{F}$ is non-symmetric.

Deformation tensors

Several rotation-independent deformation tensors are used in mechanics. In solid mechanics, the most popular of these are the right and left Cauchy-Green deformation tensors.

Since a pure rotation should not induce any stresses in a deformable body, it is often convenient to use rotation-independent measures of deformation in continuum mechanics. As a rotation followed by its inverse rotation leads to no change ($\mathbf{R}\mathbf{R}^T = \mathbf{R}^T\mathbf{R} = \mathbf{1}$) we can exclude the rotation by multiplying $\mathbf{F}$ by its transpose.

The Right Cauchy-Green deformation tensor

In 1839, George Green introduced a deformation tensor known as the *right Cauchy-Green deformation tensor* or *Green's deformation tensor*, defined as[2][3]:

$$\mathbf{C} = \mathbf{F}^T\mathbf{F} = \mathbf{U}^2 \qquad \text{or} \qquad C_{IJ} = F_{kI}\ F_{kJ} = \frac{\partial x_k}{\partial X_I}\frac{\partial x_k}{\partial X_J}.$$

Physically, the Cauchy-Green tensor gives us the square of local change in distances due to deformation, i.e. $d\mathbf{x}^2 = d\mathbf{X}\cdot\mathbf{C}d\mathbf{X}$

Invariants of $\mathbf{C}$ are often used in the expressions for strain energy density functions. The most commonly used invariants are

$$\begin{aligned}
I_1^C &:= \mathrm{tr}(\mathbf{C}) = C_{II} = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 \\
I_2^C &:= \tfrac{1}{2}\left[(\mathrm{tr}~\mathbf{C})^2 - \mathrm{tr}(\mathbf{C}^2)\right] = \tfrac{1}{2}\left[(C_{JJ})^2 - C_{IK}C_{KI}\right] = \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2 \\
I_3^C &:= \det(\mathbf{C}) = \lambda_1^2\lambda_2^2\lambda_3^2.
\end{aligned}$$

The Finger deformation tensor

The IUPAC recommends[3] that the inverse of the right Cauchy-Green deformation tensor (called the Cauchy tensor in that document), i. e., $\mathbf{C}^{-1}$, be called the **Finger tensor**. However, that nomenclature is not universally accepted in applied mechanics.

$$\mathbf{f} = \mathbf{C}^{-1} = \mathbf{F}^{-1}\mathbf{F}^{-T} \qquad \text{or} \qquad f_{IJ} = \frac{\partial X_I}{\partial x_k}\frac{\partial X_J}{\partial x_k}$$

The Left Cauchy-Green or Finger deformation tensor

Reversing the order of multiplication in the formula for the right Green-Cauchy deformation tensor leads to the *left Cauchy-Green deformation tensor* which is defined as:

$$\mathbf{B} = \mathbf{F}\mathbf{F}^T = \mathbf{V}^2 \qquad \text{or} \qquad B_{ij} = \frac{\partial x_i}{\partial X_K}\frac{\partial x_j}{\partial X_K}$$

The left Cauchy-Green deformation tensor *is often called the* Finger deformation tensor*, named after Josef Finger (1894).*[3][4][5]

Invariants of $\mathbf{B}$ are also used in the expressions for strain energy density functions. The conventional invariants are defined as

$$I_1 := \mathrm{tr}(\mathbf{B}) = B_{ii} = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$$
$$I_2 := \tfrac{1}{2}\left[(\mathrm{tr}~\mathbf{B})^2 - \mathrm{tr}(\mathbf{B}^2)\right] = \tfrac{1}{2}\left(B_{ii}^2 - B_{jk}B_{kj}\right) = \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2$$
$$I_3 := \det\mathbf{B} = J^2 = \lambda_1^2\lambda_2^2\lambda_3^2$$

where $J := \det\mathbf{F}$ is the determinant of the deformation gradient.

For nearly incompressible materials, a slightly different set of invariants is used:

$$(\bar{I}_1 := J^{-2/3} I_1 ~;~~ \bar{I}_2 := J^{-4/3} I_2 ~;~~ J = 1) ~.$$

The Cauchy deformation tensor

Earlier in 1828,[6] Augustin Louis Cauchy introduced a deformation tensor defined as the inverse of the left Cauchy-Green deformation tensor, $\mathbf{B}^{-1}$. This tensor has also been called the **Piola tensor**[3] and the **Finger tensor**[7] in the rheology and fluid dynamics literature.

$$\mathbf{c} = \mathbf{B}^{-1} = \mathbf{F}^{-T}\mathbf{F}^{-1} \qquad \text{or} \qquad c_{ij} = \frac{\partial X_K}{\partial x_i}\frac{\partial X_K}{\partial x_j}$$

Spectral representation

If there are three distinct *principal stretches* λ_i, the spectral decompositions of $\mathbf{C}$ and $\mathbf{B}$ is given by

$$\mathbf{C} = \sum_{i=1}^3 \lambda_i^2 \mathbf{N}_i \otimes \mathbf{N}_i \qquad \text{and} \qquad \mathbf{B} = \sum_{i=1}^3 \lambda_i^2 \mathbf{n}_i \otimes \mathbf{n}_i$$

Furthermore,

$$\mathbf{U} = \sum_{i=1}^3 \lambda_i \mathbf{N}_i \otimes \mathbf{N}_i ~;~~ \mathbf{V} = \sum_{i=1}^3 \lambda_i \mathbf{n}_i \otimes \mathbf{n}_i$$
$$\mathbf{R} = \sum_{i=1}^3 \mathbf{n}_i \otimes \mathbf{N}_i ~;~~ \mathbf{F} = \sum_{i=1}^3 \lambda_i \mathbf{n}_i \otimes \mathbf{N}_i$$

Observe that

$$\mathbf{V} = \mathbf{R}~\mathbf{U}~\mathbf{R}^T = \sum_{i=1}^3 \lambda_i~\mathbf{R}~(\mathbf{N}_i \otimes \mathbf{N}_i)~\mathbf{R}^T = \sum_{i=1}^3 \lambda_i~(\mathbf{R}~\mathbf{N}_i) \otimes (\mathbf{R}~\mathbf{N}_i)$$

Therefore the uniqueness of the spectral decomposition also implies that $\mathbf{n}_i = \mathbf{R}~\mathbf{N}_i$. The left stretch ($\mathbf{V}$) is also called the *spatial stretch tensor* while the right stretch ($\mathbf{U}$) is called the *material stretch tensor*.

The effect of $\mathbf{F}$ acting on $\mathbf{N}_i$ is to stretch the vector by λ_i and to rotate it to the new orientation $\mathbf{n}_i$, i.e.,

$$\mathbf{F}~\mathbf{N}_i = \lambda_i~(\mathbf{R}~\mathbf{N}_i) = \lambda_i~\mathbf{n}_i$$

In a similar vein,

$$\mathbf{F}^{-T}~\mathbf{N}_i = \frac{1}{\lambda_i}~\mathbf{n}_i ~;~~ \mathbf{F}^T~\mathbf{n}_i = \lambda_i~\mathbf{N}_i ~;~~ \mathbf{F}^{-1}~\mathbf{n}_i = \frac{1}{\lambda_i}~\mathbf{N}_i ~.$$

Examples

Uniaxial extension of an incompressible material

This is the case where a specimen is stretched in 1-direction with a stretch ratio of $\alpha = \alpha_1$. If the volume remains constant, the contraction in the other two directions is such that $\alpha_1\alpha_2\alpha_3 = 1$ or $\alpha_2 = \alpha_3 = \alpha^{-0.5}$. Then:

$$\mathbf{F} = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \alpha^{-0.5} & 0 \\ 0 & 0 & \alpha^{-0.5} \end{bmatrix}$$

$$\mathbf{B} = \mathbf{C} = \begin{bmatrix} \alpha^2 & 0 & 0 \\ 0 & \alpha^{-1} & 0 \\ 0 & 0 & \alpha^{-1} \end{bmatrix}$$

Simple shear

$$\mathbf{F} = \begin{bmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} 1+\gamma^2 & \gamma & 0 \\ \gamma & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} 1 & \gamma & 0 \\ \gamma & 1+\gamma^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Rigid body rotation

$$\mathbf{F} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{B} = \mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{1}$$

Derivatives of stretch

Derivatives of the stretch with respect to the right Cauchy-Green deformation tensor are used to derive the stress-strain relations of many solids, particularly hyperelastic materials. These derivatives are

$$\frac{\partial \lambda_i}{\partial \mathbf{C}} = \frac{1}{2\lambda_i}\,\mathbf{N}_i \otimes \mathbf{N}_i = \frac{1}{2\lambda_i}\,\mathbf{R}^T\,(\mathbf{n}_i \otimes \mathbf{n}_i)\,\mathbf{R}\ ; \quad i = 1,2,3$$

and follow from the observations that

$$\mathbf{C} : (\mathbf{N}_i \otimes \mathbf{N}_i) = \lambda_i^2\ ; \qquad \frac{\partial \mathbf{C}}{\partial \mathbf{C}} = \mathsf{I}^{(s)}\ ; \qquad \mathsf{I}^{(s)} : (\mathbf{N}_i \otimes \mathbf{N}_i) = \mathbf{N}_i \otimes \mathbf{N}_i.$$

Physical interpretation of deformation tensors

Let $\mathbf{X} = X^i\ \boldsymbol{E}_i$ be a Cartesian coordinate system defined on the undeformed body and let $\mathbf{x} = x^i\ \boldsymbol{E}_i$ be another system defined on the deformed body. Let a curve $\mathbf{X}(s)$ in the undeformed body be parametrized using $s \in [0,1]$. Its image in the deformed body is $\mathbf{x}(\mathbf{X}(s))$.
The undeformed length of the curve is given by

$$l_X = \int_0^1 \left|\frac{d\mathbf{X}}{ds}\cdot\frac{d\mathbf{X}}{ds}\right|\ ds = \int_0^1 \left|\frac{d\mathbf{X}}{ds}\cdot\boldsymbol{I}\cdot\frac{d\mathbf{X}}{ds}\right|\ ds$$

After deformation, the length becomes

$$l_x = \int_0^1 \left| \frac{d\mathbf{x}}{ds} \cdot \frac{d\mathbf{x}}{ds} \right| \, ds = \int_0^1 \left| \left(\frac{d\mathbf{x}}{d\mathbf{X}} \cdot \frac{d\mathbf{X}}{ds} \right) \cdot \left(\frac{d\mathbf{x}}{d\mathbf{X}} \cdot \frac{d\mathbf{X}}{ds} \right) \right| \, ds$$
$$= \int_0^1 \left| \frac{d\mathbf{X}}{ds} \cdot \left[\left(\frac{d\mathbf{x}}{d\mathbf{X}} \right)^T \cdot \frac{d\mathbf{x}}{d\mathbf{X}} \right] \cdot \frac{d\mathbf{X}}{ds} \right| \, ds$$

Note that the right Cauchy-Green deformation tensor is defined as

$$\boldsymbol{C} := \boldsymbol{F}^T \cdot \boldsymbol{F} = \left(\frac{d\mathbf{x}}{d\mathbf{X}} \right)^T \cdot \frac{d\mathbf{x}}{d\mathbf{X}}$$

Hence,

$$l_x = \int_0^1 \left| \frac{d\mathbf{X}}{ds} \cdot \boldsymbol{C} \cdot \frac{d\mathbf{X}}{ds} \right| \, ds$$

which indicates that changes in length are characterized by $\boldsymbol{C}$.

Finite strain tensors

The concept of *strain* is used to evaluate how much a given displacement differs locally from a rigid body displacement (Ref. Lubliner). One of such strains for large deformations is the *Lagrangian finite strain tensor*, also called the *Green-Lagrangian strain tensor* or *Green – St-Venant strain tensor*, defined as

$$\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I}) \qquad \text{or} \qquad E_{KL} = \frac{1}{2}\left(\frac{\partial x_j}{\partial X_K}\frac{\partial x_j}{\partial X_L} - \delta_{KL} \right)$$

or as a function of the displacement gradient tensor

$$\mathbf{E} = \frac{1}{2}\left[(\nabla_{\mathbf{X}}\mathbf{u})^T + \nabla_{\mathbf{X}}\mathbf{u} + (\nabla_{\mathbf{X}}\mathbf{u})^T \cdot \nabla_{\mathbf{X}}\mathbf{u} \right]$$

or

$$E_{KL} = \frac{1}{2}\left(\frac{\partial U_K}{\partial X_L} + \frac{\partial U_L}{\partial X_K} + \frac{\partial U_M}{\partial X_K}\frac{\partial U_M}{\partial X_L} \right)$$

The Green-Lagrangian strain tensor is a measure of how much $\mathbf{C}$ differs from $\mathbf{I}$. It can be shown that this tensor is a special case of a general formula for Lagrangian strain tensors (Hill 1968):

$$\mathbf{E}_{(m)} = \frac{1}{2m}(\mathbf{U}^{2m} - \mathbf{I})$$

For different values of m we have:

$$\begin{aligned} \mathbf{E}_{(1)} &= \frac{1}{2}(\mathbf{U}^2 - \mathbf{I}) \qquad \text{Green-Lagrangian strain tensor} \\ \mathbf{E}_{(1/2)} &= (\mathbf{U} - \mathbf{I}) \qquad \text{Biot strain tensor} \\ \mathbf{E}_{(0)} &= \ln \mathbf{U} \qquad \text{Logarithmic strain, Natural strain, True strain, or Hencky strain} \end{aligned}$$

The *Eulerian-Almansi finite strain tensor*, referenced to the deformed configuration, i.e. Eulerian description, is defined as

$$\mathbf{e} = \frac{1}{2}(\mathbf{I} - \mathbf{c}) \qquad \text{or} \qquad e_{rs} = \frac{1}{2}\left(\delta_{rs} - \frac{\partial X_M}{\partial x_r}\frac{\partial X_M}{\partial x_s} \right)$$

or as a function of the displacement gradients we have

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j} \right)$$

Derivation of the Lagrangian and Eulerain finite strain tensors

A measure of deformation is the difference between the squares of the differential line element $d\mathbf{X}$, in the undeformed configuration, and $d\mathbf{x}$, in the deformed configuration (Figure 2). Deformation has occurred if the difference is non zero, otherwise a rigid-body displacement has occurred. Thus we have,

$$d\mathbf{x}^2 - d\mathbf{X}^2 = d\mathbf{x}\cdot d\mathbf{x} - d\mathbf{X}\cdot d\mathbf{X} \qquad \text{or} \qquad (dx)^2-(dX)^2 = dx_j dx_j - dX_M\, dX_M$$

In the Lagrangian description, using the material coordinates as the frame of reference, the linear transformation between the differential lines is

$$d\mathbf{x} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}}\, d\mathbf{X} = \mathbf{F}\, d\mathbf{X} \qquad \text{or} \qquad dx_j = \frac{\partial x_j}{\partial X_M}\, dX_M$$

Then we have,

$$\begin{aligned} d\mathbf{x}^2 &= d\mathbf{x}\cdot d\mathbf{x} \\ &= \mathbf{F}\cdot d\mathbf{X}\cdot \mathbf{F}\cdot d\mathbf{X} \\ &= d\mathbf{X}\cdot \mathbf{F}^T\mathbf{F}\cdot d\mathbf{X} \\ &= d\mathbf{X}\cdot \mathbf{C}\cdot d\mathbf{X} \end{aligned} \qquad \text{or} \qquad \begin{aligned} (dx)^2 &= dx_j\, dx_j \\ &= \frac{\partial x_j}{\partial X_K}\frac{\partial x_j}{\partial X_L}\, dX_K\, dX_L \\ &= C_{KL}\, dX_K\, dX_L \end{aligned}$$

where C_{KL} are the components of the *right Cauchy-Green deformation tensor*, $\mathbf{C} = \mathbf{F}^T\mathbf{F}$. Then, replacing this equation into the first equation we have,

$$\begin{aligned} d\mathbf{x}^2 - d\mathbf{X}^2 &= d\mathbf{X}\cdot \mathbf{C}\cdot d\mathbf{X} - d\mathbf{X}\cdot d\mathbf{X} \\ &= d\mathbf{X}\cdot(\mathbf{C}-\mathbf{I})\cdot d\mathbf{X} \\ &= d\mathbf{X}\cdot 2\mathbf{E}\cdot d\mathbf{X} \end{aligned}$$

or

$$\begin{aligned} (dx)^2-(dX)^2 &= \frac{\partial x_j}{\partial X_K}\frac{\partial x_j}{\partial X_L}\, dX_K\, dX_L - dX_M\, dX_M \\ &= \left(\frac{\partial x_j}{\partial X_K}\frac{\partial x_j}{\partial X_L} - \delta_{KL}\right) dX_K\, dX_L \\ &= 2E_{KL}\, dX_K\, dX_L \end{aligned}$$

where E_{KL}, are the components of a second-order tensor called the *Green – St-Venant strain tensor* or the *Lagrangian finite strain tensor*,

$$\mathbf{E} = \frac{1}{2}(\mathbf{C}-\mathbf{I}) \qquad \text{or} \qquad E_{KL} = \frac{1}{2}\left(\frac{\partial x_j}{\partial X_K}\frac{\partial x_j}{\partial X_L} - \delta_{KL}\right)$$

In the Eulerian description, using the spatial coordinates as the frame of reference, the linear transformation between the differential lines is

$$d\mathbf{X} = \frac{\partial \mathbf{X}}{\partial \mathbf{x}}d\mathbf{x} = \mathbf{F}^{-1}\, d\mathbf{x} = \mathbf{H}\, d\mathbf{x} \qquad \text{or} \qquad dX_M = \frac{\partial X_M}{\partial x_n}\, dx_n$$

where $\frac{\partial X_M}{\partial x_n}$ are the components of the *spatial deformation gradient tensor*, $\mathbf{H}$. Thus we have

$$\begin{aligned} d\mathbf{X}^2 &= d\mathbf{X}\cdot d\mathbf{X} \\ &= \mathbf{F}^{-1}\cdot d\mathbf{x}\cdot \mathbf{F}^{-1}\cdot d\mathbf{x} \\ &= d\mathbf{x}\cdot \mathbf{F}^{-T}\mathbf{F}^{-1}\cdot d\mathbf{x} \\ &= d\mathbf{x}\cdot \mathbf{c}\cdot d\mathbf{x} \end{aligned} \qquad \text{or} \qquad \begin{aligned} (dX)^2 &= dX_M\, dX_M \\ &= \frac{\partial X_M}{\partial x_r}\frac{\partial X_M}{\partial x_s}\, dx_r\, dx_s \\ &= c_{rs}\, dx_r\, dx_s \end{aligned}$$

where the second order tensor c_{rs} is called *Cauchy's deformation tensor*, $\mathbf{c} = \mathbf{F}^{-T}\mathbf{F}^{-1}$. Then we have,

$$\begin{aligned} d\mathbf{x}^2 - d\mathbf{X}^2 &= d\mathbf{x}\cdot d\mathbf{x} - d\mathbf{x}\cdot \mathbf{c}\cdot d\mathbf{x} \\ &= d\mathbf{x}\cdot(\mathbf{I}-\mathbf{c})\cdot d\mathbf{x} \\ &= d\mathbf{x}\cdot 2\mathbf{e}\cdot d\mathbf{x} \end{aligned}$$

or

$$\begin{aligned}(dx)^2 - (dX)^2 &= dx_j\,dx_j - \frac{\partial X_M}{\partial x_r}\frac{\partial X_M}{\partial x_s}\,dx_r\,dx_s \\ &= \left(\delta_{rs} - \frac{\partial X_M}{\partial x_r}\frac{\partial X_M}{\partial x_s}\right)\,dx_r\,dx_s \\ &= 2e_{rs}\,dx_r\,dx_s\end{aligned}$$

where e_{rs}, are the components of a second-order tensor called the *Eulerian-Almansi finite strain tensor*,

$$\mathbf{e} = \frac{1}{2}(\mathbf{I} - \mathbf{c}) \qquad \text{or} \qquad e_{rs} = \frac{1}{2}\left(\delta_{rs} - \frac{\partial X_M}{\partial x_r}\frac{\partial X_M}{\partial x_s}\right)$$

Both the Lagrangian and Eulerian finite strain tensors can be conveniently expressed in terms of the *displacement gradient tensor*. For the Lagrangian strain tensor, first we differentiate the displacement vector $\mathbf{u}(\mathbf{X},t)$ with respect to the material coordinates X_M to obtain the *material displacement gradient tensor*, $\nabla_{\mathbf{X}}\mathbf{u}$

$$\begin{aligned}\mathbf{u}(\mathbf{X},t) &= \mathbf{x}(\mathbf{X},t) - \mathbf{X} \\ \nabla_{\mathbf{X}}\mathbf{u} &= \mathbf{F} - \mathbf{I} \\ \mathbf{F} &= \nabla_{\mathbf{X}}\mathbf{u} + \mathbf{I}\end{aligned} \qquad \text{or} \qquad \begin{aligned}u_i &= x_i - \delta_{iJ}X_J \\ \delta_{iJ}U_J &= x_i - \delta_{iJ}X_J \\ x_i &= \delta_{iJ}\left(U_J + X_J\right) \\ \frac{\partial x_i}{\partial X_K} &= \delta_{iJ}\left(\frac{\partial U_J}{\partial X_K} + \delta_{JK}\right)\end{aligned}$$

Replacing this equation into the expression for the Lagrangian finite strain tensor we have

$$\begin{aligned}\mathbf{E} &= \frac{1}{2}\left(\mathbf{F}^T\mathbf{F} - \mathbf{I}\right) \\ &= \frac{1}{2}\left[\left\{(\nabla_{\mathbf{X}}\mathbf{u})^T + \mathbf{I}\right\}(\nabla_{\mathbf{X}}\mathbf{u} + \mathbf{I}) - \mathbf{I}\right] \\ &= \frac{1}{2}\left[(\nabla_{\mathbf{X}}\mathbf{u})^T + \nabla_{\mathbf{X}}\mathbf{u} + (\nabla_{\mathbf{X}}\mathbf{u})^T\cdot\nabla_{\mathbf{X}}\mathbf{u}\right]\end{aligned}$$

or

$$\begin{aligned}E_{KL} &= \frac{1}{2}\left(\frac{\partial x_j}{\partial X_K}\frac{\partial x_j}{\partial X_L} - \delta_{KL}\right) \\ &= \frac{1}{2}\left[\delta_{jM}\left(\frac{\partial U_M}{\partial X_K} + \delta_{MK}\right)\delta_{jN}\left(\frac{\partial U_N}{\partial X_L} + \delta_{NL}\right) - \delta_{KL}\right] \\ &= \frac{1}{2}\left[\delta_{MN}\left(\frac{\partial U_M}{\partial X_K} + \delta_{MK}\right)\left(\frac{\partial U_N}{\partial X_L} + \delta_{NL}\right) - \delta_{KL}\right] \\ &= \frac{1}{2}\left[\left(\frac{\partial U_M}{\partial X_K} + \delta_{MK}\right)\left(\frac{\partial U_M}{\partial X_L} + \delta_{ML}\right) - \delta_{KL}\right] \\ &= \frac{1}{2}\left(\frac{\partial U_K}{\partial X_L} + \frac{\partial U_L}{\partial X_K} + \frac{\partial U_M}{\partial X_K}\frac{\partial U_M}{\partial X_L}\right)\end{aligned}$$

Similarly, the Eulerian-Almansi finite strain tensor can be expressed as

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}\right)$$

Stretch ratio

The **stretch ratio** is a measure of the extensional or normal strain of a differential line element, which can be defined at either the undeformed configuration or the deformed configuration.

The stretch ratio for the differential element $d\mathbf{X} = dX\mathbf{N}$ (Figure) in the direction of the unit vector $\mathbf{N}$ at the material point P, in the undeformed configuration, is defined as

$$\Lambda_{(\mathbf{N})} = \frac{dx}{dX}$$

where dx is the deformed magnitude of the differential element $d\mathbf{X}$.

Similarly, the stretch ratio for the differential element $d\mathbf{x} = dx\mathbf{n}$ (Figure), in the direction of the unit vector $\mathbf{n}$ at the material point p, in the deformed configuration, is defined as

$$\frac{1}{\Lambda_{(\mathbf{n})}} = \frac{dX}{dx}.$$

The normal strain $e_{\mathbf{N}}$ in any direction $\mathbf{N}$ can be expressed as a function of the stretch ratio,

$$e_{(\mathbf{N})} = \frac{dx - dX}{dX} = \Lambda_{(\mathbf{N})} - 1.$$

This equation implies that the normal strain is zero, i.e. no deformation, when the stretch is equal to unity. Some materials, such as elastometers can sustain stretch ratios of 3 or 4 before they fail, whereas traditional engineering materials, such as concrete or steel, fail at much lower stretch ratios, perhaps of the order of 1.001 (reference?)

Physical interpretation of the finite strain tensor

The diagonal components E_{KL} of the Lagrangian finite strain tensor are related to the normal strain, e.g.

$$E_{11} = e_{(\mathbf{I}_1)} + \frac{1}{2}e^2_{(\mathbf{I}_1)}$$

where $e_{(\mathbf{I}_1)}$ is the normal strain or engineering strain in the direction $\mathbf{I}_1$.

The off-diagonal components E_{KL} of the Lagrangian finite strain tensor are related to shear strain, e.g.

$$E_{12} = \frac{1}{2}\sqrt{2E_{11}+1}\sqrt{2E_{22}+1}\sin\phi_{12}$$

where ϕ_{12} is the change in the angle between two line elements that were originally perpendicular with directions $\mathbf{I}_1$ and $\mathbf{I}_2$, respectively.

Under certain circumstances, i.e. small displacements and small displacement rates, the components of the Lagrangian finite strain tensor may be approximated by the components of the infinitesimal strain tensor

Derivation of the physical interpretation of the Lagrangian and Eulerian finite strain tensors

The stretch ratio for the differential element $d\mathbf{X} = dX\mathbf{N}$ (Figure) in the direction of the unit vector $\mathbf{N}$ at the material point P, in the undeformed configuration, is defined as

$$\Lambda_{(\mathbf{N})} = \frac{dx}{dX}$$

where dx is the deformed magnitude of the differential element $d\mathbf{X}$.

Similarly, the stretch ratio for the differential element $d\mathbf{x} = dx\mathbf{n}$ (Figure), in the direction of the unit vector $\mathbf{n}$ at the material point p, in the deformed configuration, is defined as

$$\frac{1}{\Lambda_{(\mathbf{n})}} = \frac{dX}{dx}$$

The square of the stretch ratio is defined as

$$\Lambda_{(\mathbf{N})}^2 = \left(\frac{dx}{dX}\right)^2$$

Knowing that

$$(dx)^2 = C_{KL} dX_K dX_L$$

we have

$$\Lambda_{(\mathbf{N})}^2 = C_{KL} N_K N_L$$

where N_K and N_L are unit vectors.

The normal strain or engineering strain $e_{\mathbf{N}}$ in any direction $\mathbf{N}$ can be expressed as a function of the stretch ratio,

$$e_{(\mathbf{N})} = \frac{dx - dX}{dX} = \Lambda_{(\mathbf{N})} - 1$$

Thus, the normal strain in the direction $\mathbf{I}_1$ at the material point P may be expressed in terms of the stretch ratio as

$$\begin{aligned} e_{(\mathbf{I}_1)} = \frac{dx_1 - dX_1}{dX_1} &= \Lambda_{(\mathbf{I}_1)} - 1 \\ &= \sqrt{C_{11}} - 1 = \sqrt{\delta_{11} + 2E_{11}} - 1 \\ &= \sqrt{1 + 2E_{11}} - 1 \end{aligned}$$

solving for E_{11} we have

$$\begin{aligned} 2E_{11} &= \frac{(dx_1)^2 - (dX_1)^2}{(dX_1)^2} \\ E_{11} &= \left(\frac{dx_1 - dX_1}{dX_1}\right) + \frac{1}{2}\left(\frac{dx_1 - dX_1}{dX_1}\right)^2 \\ &= e_{(\mathbf{I}_1)} + \frac{1}{2} e_{(\mathbf{I}_1)}^2 \end{aligned}$$

The *shear strain*, or change in angle between two line elements $d\mathbf{X}_1$ and $d\mathbf{X}_2$ initially perpendicular, and oriented in the principal directions $\mathbf{I}_1$ and $\mathbf{I}_2$, respectivelly, can also be expressed as a function of the stretch ratio. From the dot product between the deformed lines $d\mathbf{x}_1$ and $d\mathbf{x}_2$ we have

$$\begin{aligned} d\mathbf{x}_1 \cdot d\mathbf{x}_2 &= dx_1 dx_2 \cos\theta_{12} \\ \mathbf{F} \cdot d\mathbf{X}_1 \cdot \mathbf{F} \cdot d\mathbf{X}_2 &= \sqrt{d\mathbf{X}_1 \cdot \mathbf{F}^T \cdot \mathbf{F} \cdot d\mathbf{X}_1} \cdot \sqrt{d\mathbf{X}_2 \cdot \mathbf{F}^T \cdot \mathbf{F} \cdot d\mathbf{X}_2} \cos\theta_{12} \\ \frac{d\mathbf{X}_1 \cdot \mathbf{F}^T \cdot \mathbf{F} \cdot d\mathbf{X}_2}{dX_1 dX_2} &= \frac{\sqrt{d\mathbf{X}_1 \cdot \mathbf{F}^T \cdot \mathbf{F} \cdot d\mathbf{X}_1} \cdot \sqrt{d\mathbf{X}_2 \cdot \mathbf{F}^T \cdot \mathbf{F} \cdot d\mathbf{X}_2}}{dX_1 dX_2} \cos\theta_{12} \\ \mathbf{I}_1 \cdot \mathbf{C} \cdot \mathbf{I}_2 &= \Lambda_{\mathbf{I}_1} \Lambda_{\mathbf{I}_2} \cos\theta_{12} \end{aligned}$$

where θ_{12} is the angle between the lines $d\mathbf{x}_1$ and $d\mathbf{x}_2$ in the deformed configuration. Defining ϕ_{12} as the shear strain or reduction in the angle between two line elements that were originally perpendicular, we have

$$\phi_{12} = \frac{\pi}{2} - \theta_{12}$$

thus,

$$\cos\theta_{12} = \sin\phi_{12}$$

then

$$\mathbf{I}_1 \cdot \mathbf{C} \cdot \mathbf{I}_2 = \Lambda_{\mathbf{I}_1} \Lambda_{\mathbf{I}_2} \sin\phi_{12}$$

or

$$\begin{aligned}
C_{12} &= \sqrt{C_{11}}\sqrt{C_{22}}\sin\phi_{12} \\
2E_{12}+\delta_{12} &= \sqrt{2E_{11}+1}\sqrt{2E_{22}+1}\sin\phi_{12} \\
E_{12} &= \frac{1}{2}\sqrt{2E_{11}+1}\sqrt{2E_{22}+1}\sin\phi_{12}
\end{aligned}$$

Deformation tensors in curvilinear coordinates

A representation of deformation tensors in curvilinear coordinates is useful for many problems in continuum mechanics such as nonlinear shell theories and large plastic deformations. Let $\mathbf{x} = \mathbf{x}(\xi^1, \xi^2, \xi^3)$ be a given deformation where the space is characterized by the coordinates (ξ^1, ξ^2, ξ^3). The tangent vector to the coordinate curve ξ^i at $\mathbf{x}$ is given by

$$\mathbf{g}_i = \frac{\partial \mathbf{x}}{\partial \xi^i}$$

The three tangent vectors at $\mathbf{x}$ form a basis. These vectors are related the reciprocal basis vectors by

$$\mathbf{g}_i \cdot \mathbf{g}^j = \delta_i^j$$

Let us define a field

$$g_{ij} := \frac{\partial \mathbf{x}}{\partial \xi^i} \cdot \frac{\partial \mathbf{x}}{\partial \xi^j} = \mathbf{g}_i \cdot \mathbf{g}_j$$

The Christoffel symbols of the first kind can be expressed as

$$\Gamma_{ijk} = \tfrac{1}{2}[(\mathbf{g}_i \cdot \mathbf{g}_k)_{,j} + (\mathbf{g}_j \cdot \mathbf{g}_k)_{,i} - (\mathbf{g}_i \cdot \mathbf{g}_j)_{,k}]$$

To see how the Christoffel symbols are related to the Right Cauchy-Green deformation tensor let us define two sets of bases

$$\mathbf{G}_i := \frac{\partial \mathbf{X}}{\partial \xi^i} ~;~~ \mathbf{G}_i \cdot \mathbf{G}^j = \delta_i^j ~;~~ \mathbf{g}_i := \frac{\partial \mathbf{x}}{\partial \xi^i} ~;~~ \mathbf{g}_i \cdot \mathbf{g}^j = \delta_i^j$$

The deformation gradient in curvilinear coordinates

Using the definition of the gradient of a vector field in curvilinear coordinates, the deformation gradient can be written as

$$\boldsymbol{F} = \boldsymbol{\nabla}_{\mathbf{X}}\mathbf{x} = \frac{\partial \mathbf{x}}{\partial \xi^i} \otimes \mathbf{G}^i = \mathbf{g}_i \otimes \mathbf{G}^i$$

The right Cauchy-Green tensor in curvilinear coordinates

The right Cauchy-Green deformation tensor is given by

$$\boldsymbol{C} = \boldsymbol{F}^T \cdot \boldsymbol{F} = (\mathbf{G}^i \otimes \mathbf{g}_i) \cdot (\mathbf{g}_i \otimes \mathbf{G}^i) = (\mathbf{g}_i \cdot \mathbf{g}_j)(\mathbf{G}^i \otimes \mathbf{G}^j)$$

If we express $\boldsymbol{C}$ in terms of components with respect to the basis { $\mathbf{G}^i$ } we have

$$\boldsymbol{C} = C_{ij}~\mathbf{G}^i \otimes \mathbf{G}^j$$

Therefore

$$C_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j = g_{ij}$$

and the Christoffel symbol of the first kind may be written in the following form.

$$\Gamma_{ijk} = \tfrac{1}{2}[C_{ik,j} + C_{jk,i} - C_{ij,k}] = \tfrac{1}{2}[(\mathbf{G}_i \cdot \boldsymbol{C} \cdot \mathbf{G}_k)_{,j} + (\mathbf{G}_j \cdot \boldsymbol{C} \cdot \mathbf{G}_k)_{,i} - (\mathbf{G}_i \cdot \boldsymbol{C} \cdot \mathbf{G}_j)_{,k}]$$

Some relations between deformation measures and Christoffel symbols

Let us consider a one-to-one mapping from $\mathbf{X} = \{X^1, X^2, X^3\}$ to $\mathbf{x} = \{x^1, x^2, x^3\}$ and let us assume that there exist two positive definite, symmetric second-order tensor fields $\boldsymbol{G}$ and $\boldsymbol{g}$ that satisfy

$$G_{ij} = \frac{\partial X^\alpha}{\partial x^i}~\frac{\partial X^\beta}{\partial x^j}~g_{\alpha\beta}$$

Then,

$$\frac{\partial G_{ij}}{\partial x^k} = \left(\frac{\partial^2 X^\alpha}{\partial x^i \partial x^k}~\frac{\partial X^\beta}{\partial x^j} + \frac{\partial X^\alpha}{\partial x^i}~\frac{\partial^2 X^\beta}{\partial x^j \partial x^k}\right)~g_{\alpha\beta} + \frac{\partial X^\alpha}{\partial x^i}~\frac{\partial X^\beta}{\partial x^j}~\frac{\partial g_{\alpha\beta}}{\partial x^k}$$

Noting that

$$\frac{\partial g_{\alpha\beta}}{\partial x^k} = \frac{\partial X^\gamma}{\partial x^k}~\frac{\partial g_{\alpha\beta}}{\partial X^\gamma}$$

and $g_{\alpha\beta} = g_{\beta\alpha}$ we have

$$\begin{aligned}
\frac{\partial G_{ij}}{\partial x^k} & = \left(\frac{\partial^2 X^\alpha}{\partial x^i \partial x^k}~\frac{\partial X^\beta}{\partial x^j} + \frac{\partial^2 X^\alpha}{\partial x^j \partial x^k}~\frac{\partial X^\beta}{\partial x^i}\right)~g_{\alpha\beta} + \frac{\partial X^\alpha}{\partial x^i}~\frac{\partial X^\beta}{\partial x^j}~\frac{\partial X^\gamma}{\partial x^k}~\frac{\partial g_{\alpha\beta}}{\partial X^\gamma} \\
\frac{\partial G_{ik}}{\partial x^j} & = \left(\frac{\partial^2 X^\alpha}{\partial x^i \partial x^j}~\frac{\partial X^\beta}{\partial x^k} + \frac{\partial^2 X^\alpha}{\partial x^j \partial x^k}~\frac{\partial X^\beta}{\partial x^i}\right)~g_{\alpha\beta} + \frac{\partial X^\alpha}{\partial x^i}~\frac{\partial X^\beta}{\partial x^k}~\frac{\partial X^\gamma}{\partial x^j}~\frac{\partial g_{\alpha\beta}}{\partial X^\gamma} \\
\frac{\partial G_{jk}}{\partial x^i} & = \left(\frac{\partial^2 X^\alpha}{\partial x^i \partial x^j}~\frac{\partial X^\beta}{\partial x^k} + \frac{\partial^2 X^\alpha}{\partial x^i \partial x^k}~\frac{\partial X^\beta}{\partial x^j}\right)~g_{\alpha\beta} + \frac{\partial X^\alpha}{\partial x^j}~\frac{\partial X^\beta}{\partial x^k}~\frac{\partial X^\gamma}{\partial x^i}~\frac{\partial g_{\alpha\beta}}{\partial X^\gamma}
\end{aligned}$$

Define

$$\begin{aligned}
{}_{(x)}\Gamma_{ijk} & := \frac{1}{2}\left(\frac{\partial G_{ik}}{\partial x^j} + \frac{\partial G_{jk}}{\partial x^i} - \frac{\partial G_{ij}}{\partial x^k}\right) \\
{}_{(X)}\Gamma_{\alpha\beta\gamma} & := \frac{1}{2}\left(\frac{\partial g_{\alpha\gamma}}{\partial X^\beta} + \frac{\partial g_{\beta\gamma}}{\partial X^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial X^\gamma}\right)
\end{aligned}$$

Hence

$${}_{(x)}\Gamma_{ijk} = \frac{\partial X^\alpha}{\partial x^i}~\frac{\partial X^\beta}{\partial x^j}~\frac{\partial X^\gamma}{\partial x^k}~{}_{(X)}\Gamma_{\alpha\beta\gamma} + \frac{\partial^2 X^\alpha}{\partial x^i \partial x^j}~\frac{\partial X^\beta}{\partial x^k}~g_{\alpha\beta}$$

Define

$$[G^{ij}] = [G_{ij}]^{-1}~;~~[g^{\alpha\beta}] = [g_{\alpha\beta}]^{-1}$$

Then

$$G^{ij} = \frac{\partial x^i}{\partial X^\alpha}~\frac{\partial x^j}{\partial X^\beta}~g^{\alpha\beta}$$

Define the Christoffel symbols of the second kind as

$${}_{(x)}\Gamma^m_{ij} := G^{mk}~{}_{(x)}\Gamma_{ijk}~;~~{}_{(X)}\Gamma^\nu_{\alpha\beta} := g^{\nu\gamma}~{}_{(X)}\Gamma_{\alpha\beta\gamma}$$

Then

$$
\begin{aligned}
{}_{(x)}\Gamma^m_{ij} &= G^{mk}\,\frac{\partial X^\alpha}{\partial x^i}\,\frac{\partial X^\beta}{\partial x^j}\,\frac{\partial X^\gamma}{\partial x^k}\,{}_{(X)}\Gamma_{\alpha\beta\gamma} + G^{mk}\,\frac{\partial^2 X^\alpha}{\partial x^i \partial x^j}\,\frac{\partial X^\beta}{\partial x^k}\,g_{\alpha\beta} \\
&= \frac{\partial x^m}{\partial X^\nu}\,\frac{\partial x^k}{\partial X^\rho}\,g^{\nu\rho}\,\frac{\partial X^\alpha}{\partial x^i}\,\frac{\partial X^\beta}{\partial x^j}\,\frac{\partial X^\gamma}{\partial x^k}\,{}_{(X)}\Gamma_{\alpha\beta\gamma} + \frac{\partial x^m}{\partial X^\nu}\,\frac{\partial x^k}{\partial X^\rho}\,g^{\nu\rho}\,\frac{\partial^2 X^\alpha}{\partial x^i \partial x^j}\,\frac{\partial X^\beta}{\partial x^k}\,g_{\alpha\beta} \\
&= \frac{\partial x^m}{\partial X^\nu}\,\delta^\gamma_\rho\,g^{\nu\rho}\,\frac{\partial X^\alpha}{\partial x^i}\,\frac{\partial X^\beta}{\partial x^j}\,{}_{(X)}\Gamma_{\alpha\beta\gamma} + \frac{\partial x^m}{\partial X^\nu}\,\delta^\beta_\rho\,g^{\nu\rho}\,\frac{\partial^2 X^\alpha}{\partial x^i \partial x^j}\,g_{\alpha\beta} \\
&= \frac{\partial x^m}{\partial X^\nu}\,g^{\nu\gamma}\,\frac{\partial X^\alpha}{\partial x^i}\,\frac{\partial X^\beta}{\partial x^j}\,{}_{(X)}\Gamma_{\alpha\beta\gamma} + \frac{\partial x^m}{\partial X^\nu}\,g^{\nu\beta}\,\frac{\partial^2 X^\alpha}{\partial x^i \partial x^j}\,g_{\alpha\beta} \\
&= \frac{\partial x^m}{\partial X^\nu}\,\frac{\partial X^\alpha}{\partial x^i}\,\frac{\partial X^\beta}{\partial x^j}\,{}_{(X)}\Gamma^\nu_{\alpha\beta} + \frac{\partial x^m}{\partial X^\nu}\,\delta^\nu_\alpha\,\frac{\partial^2 X^\alpha}{\partial x^i \partial x^j}
\end{aligned}
$$

Therefore

$$
{}_{(x)}\Gamma^m_{ij} = \frac{\partial x^m}{\partial X^\nu}\,\frac{\partial X^\alpha}{\partial x^i}\,\frac{\partial X^\beta}{\partial x^j}\,{}_{(X)}\Gamma^\nu_{\alpha\beta} + \frac{\partial x^m}{\partial X^\alpha}\,\frac{\partial^2 X^\alpha}{\partial x^i \partial x^j}
$$

The invertibility of the mapping implies that

$$
\begin{aligned}
\frac{\partial X^\mu}{\partial x^m}\,{}_{(x)}\Gamma^m_{ij} &= \frac{\partial X^\mu}{\partial x^m}\,\frac{\partial x^m}{\partial X^\nu}\,\frac{\partial X^\alpha}{\partial x^i}\,\frac{\partial X^\beta}{\partial x^j}\,{}_{(X)}\Gamma^\nu_{\alpha\beta} + \frac{\partial X^\mu}{\partial x^m}\,\frac{\partial x^m}{\partial X^\alpha}\,\frac{\partial^2 X^\alpha}{\partial x^i \partial x^j} \\
&= \delta^\mu_\nu\,\frac{\partial X^\alpha}{\partial x^i}\,\frac{\partial X^\beta}{\partial x^j}\,{}_{(X)}\Gamma^\nu_{\alpha\beta} + \delta^\mu_\alpha\,\frac{\partial^2 X^\alpha}{\partial x^i \partial x^j} \\
&= \frac{\partial X^\alpha}{\partial x^i}\,\frac{\partial X^\beta}{\partial x^j}\,{}_{(X)}\Gamma^\mu_{\alpha\beta} + \frac{\partial^2 X^\mu}{\partial x^i \partial x^j}
\end{aligned}
$$

We can also formulate a similar result in terms of derivatives with respect to x . Therefore

$$
\begin{aligned}
\frac{\partial^2 X^\mu}{\partial x^i \partial x^j} &= \frac{\partial X^\mu}{\partial x^m}\,{}_{(x)}\Gamma^m_{ij} - \frac{\partial X^\alpha}{\partial x^i}\,\frac{\partial X^\beta}{\partial x^j}\,{}_{(X)}\Gamma^\mu_{\alpha\beta} \\
\frac{\partial^2 x^m}{\partial X^\alpha \partial X^\beta} &= \frac{\partial x^m}{\partial X^\mu}\,{}_{(X)}\Gamma^\mu_{\alpha\beta} - \frac{\partial x^i}{\partial X^\alpha}\,\frac{\partial x^j}{\partial X^\beta}\,{}_{(x)}\Gamma^m_{ij}
\end{aligned}
$$

Compatibility conditions

The problem of compatibility in continuum mechanics involves the determination of allowable single-valued continuous fields on bodies. These allowable conditions leave the body without unphysical gaps or overlaps after a deformation. Most such conditions apply to simply-connected bodies. Additional conditions are required for the internal boundaries of multiply connected bodies.

Compatibility of the deformation gradient

The necessary and sufficient conditions for the existence of a compatible $\boldsymbol{F}$ field over a simply connected body are

$$
\boldsymbol{\nabla} \times \boldsymbol{F} = \boldsymbol{0}
$$

Compatibility of the right Cauchy-Green deformation tensor

The necessary and sufficient conditions for the existence of a compatible $\boldsymbol{C}$ field over a simply connected body are

$$R^\gamma_{\alpha\beta\rho} := \frac{\partial}{\partial X^\rho}[\,_{(X)}\Gamma^\gamma_{\alpha\beta}] - \frac{\partial}{\partial X^\beta}[\,_{(X)}\Gamma^\gamma_{\alpha\rho}] + \,_{(X)}\Gamma^\gamma_{\mu\rho}\,_{(X)}\Gamma^\mu_{\alpha\beta} - \,_{(X)}\Gamma^\gamma_{\mu\beta}\,_{(X)}\Gamma^\mu_{\alpha\rho} = 0$$

We can show these are the mixed components of the Riemann-Christoffel curvature tensor. Therefore the necessary conditions for $\boldsymbol{C}$ -compatibility are that the Riemann-Christoffel curvature of the deformation is zero.

Compatibility of the left Cauchy-Green deformation tensor

No general sufficiency conditions are known for the left Cauchy-Green deformation tensor in three-dimensions. Compatibility conditions for two-dimensional $\boldsymbol{B}$ fields have been found by Janet Blume.[8][9]

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External links

- Prof. Amit Acharya's notes on compatibility on iMechanica (http://www.imechanica.org/node/3786)

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