11. 12. 2020 **Številska zaporedja**

$n \in \mathbb{N}$ vzamemo $a_n \in \mathbb{R}$ $\Rightarrow$ zaporedje

$(a_1, a_2, a_3, \dots)$ $\quad (a_n)_{n \in \mathbb{N}}$

Primer 1. $a_n = a$ za vsak n, $a \in \mathbb{R}$

$(a, a, a, \dots)$ konstantno zaporedje

2. $a_n = \dfrac{(-1)^n}{n^2}$, $n \in \mathbb{N}$, $a_1 = -1$, $a_2 = \dfrac{1}{4}, \dots$

$\left(-1, \dfrac{1}{4}, -\dfrac{1}{9}, \dfrac{1}{16}, \dots\right)$

V 1. in 2: eksplicitno podajanje: formula za a_n.

3. $a_1 = 1$, $a_2 = 1$, $a_{n+2} = a_{n+1} + a_n$, $n \geq 1$

rekurzivna formula

Rekurzivno podano zaporedje

$a_3 = a_2 + a_1 = 2$

$a_4 = a_3 + a_2 = 3, \dots$ $\quad a_{10} = a_9 + a_8$

$a_5 = a_4 + a_3 = 3 + 2 = 5$

Opomba: Fibonaccijevo zaporedje

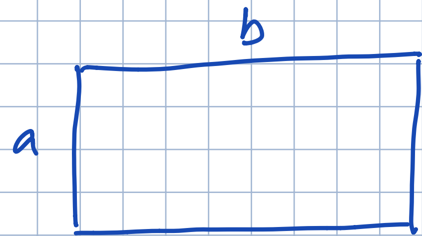
Eksplicitna formula:

$$a_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$$

$$a_1 = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}\right) = \frac{1}{\sqrt{5}}\left(\frac{\sqrt{5}}{2} + \frac{\sqrt{5}}{2}\right) = 1$$

$\alpha = \dfrac{1+\sqrt{5}}{2}$ $\rightarrow$ zlati rez.

$\dfrac{a}{b} = \dfrac{b}{a+b}$



Def. $(a_n)_{n\in\mathbb{N}}$ je monotono naraščajoče (oz. padajoče), če:

$a_{n+1} \geq a_n$ (oz. $a_{n+1} \leq a_n$)

za vsak n.

npr. $a_n = \frac{1}{n}$ $\quad 1, \frac{1}{2}, \frac{1}{3}, \ldots$

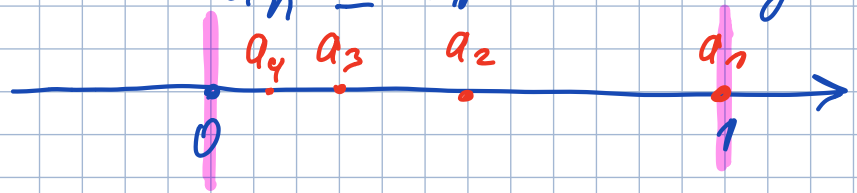
$\frac{1}{n+1} < \frac{1}{n} \iff n < n+1$ ✓

$(a_n)_{n\in\mathbb{N}}$ je padajoče.

Def. $(a_n)_{n\in\mathbb{N}}$ je navzgor (oz. navzdol) omejeno, če obstaja $M \in \mathbb{R}$, da velja $a_n \leq M$ za vsak n. M je zgornja meja
(oz. če obstaja $m \in \mathbb{R}$, da velja:
$a_n \geq m$ za vsak n)
$\nwarrow$ spodnja meja

Primer $a_n = \frac{1}{n}$ $\quad a_n > 0$ $\quad m = 0$ je spodnja meja

$a_n \leq a_1 = 1 \rightarrow$ zgornja meja



Zaporedje $(a_n)_{n\in\mathbb{N}}$ je omejeno, če je omejeno navzgor in navzdol. $a_n = \frac{1}{n}, n \in \mathbb{N}$, je omejeno.

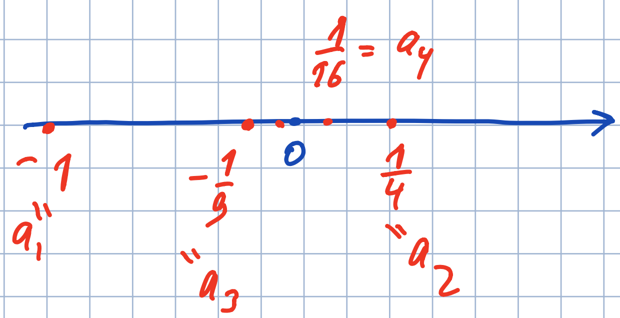
Primer: $a_n = n$. $\quad 1, 2, 3, 4, \ldots$ neomejeno, navzdol omejeno

14. 12. 2020

Limita zaporedja

Ideja: $a_n = \frac{(-1)^n}{n^2}, \quad n \geq 1$

$-1, \frac{1}{4}, -\frac{1}{9}, \frac{1}{16}, -\frac{1}{25}, \ldots$

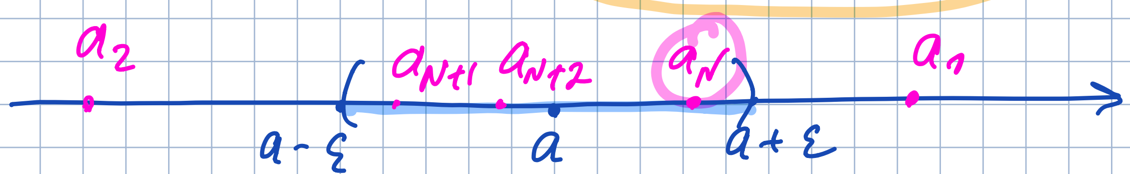


$a_{100} = \frac{1}{100^2} = \frac{1}{10000}$

Pišemo: $\lim\limits_{n \to \infty} a_n = 0$

Pozor: $a_n \neq 0$ za vsak n!

Definicija Realno število a je limita zaporedja $(a_n)_{n \in \mathbb{N}}$, če za vsak $\varepsilon > 0$ obstaja $N = N_\varepsilon$: za vsak $n \geq N_\varepsilon$: $|a_n - a| < \varepsilon$

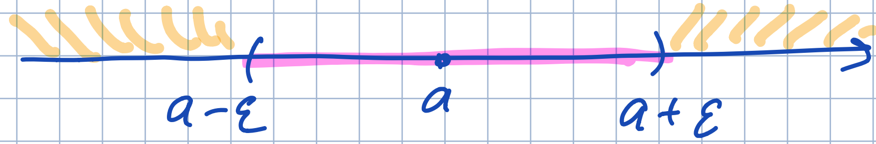


$(a-\varepsilon, a+\varepsilon)$ je ε-okolica števila a.

$\underbrace{|a_n - a|}_{\text{razdalja med } a \text{ in } a_n} < \varepsilon \iff a_n \in (a-\varepsilon, a+\varepsilon)$

Definicija pore: 1) Za vsako ε-okolico $(a-\varepsilon, a+\varepsilon)$ obstaja $N \in \mathbb{N}$, da so vsi členi zaporedja od a_N dalje v ε-okolici števila a.

2) V okolici $(a-\varepsilon, a+\varepsilon)$ ležijo vsi členi zaporedja, razen morda nekaj začetnih členov.



izven

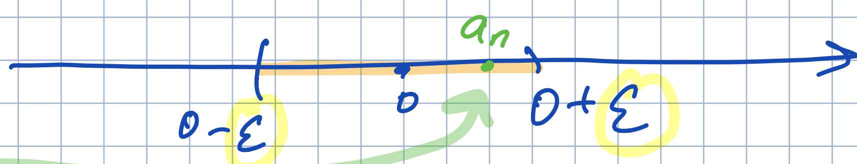
ε-okolice:

končno mnogo členov!

Primer 1 Pokažemo, da je $\lim\limits_{n\to\infty} \frac{(-1)^n}{n^2} = 0$

$\forall \varepsilon > 0$

↳ "za vsak"



$|a_n - a| < \varepsilon$ $\quad a = 0$

$|a_n| < \varepsilon$

$\left|\frac{(-1)^n}{n^2}\right| < \varepsilon \qquad \frac{1}{n^2} < \varepsilon \qquad 1 < \varepsilon n^2$

$\varepsilon n^2 > 1 \qquad n^2 > \frac{1}{\varepsilon} \qquad n > \frac{1}{\sqrt{\varepsilon}}$

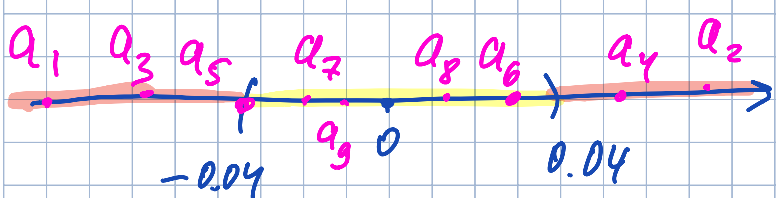
$\Rightarrow \lim\limits_{n\to\infty} a_n = 0$

Primer 2 Naj bo $\varepsilon = 0.04$. Od katerega n dalje so členi zaporedja v okolici $(-0.04, 0.04)$

$n > \frac{1}{\sqrt{\varepsilon}} \qquad n > \frac{1}{\sqrt{0.04}}$

$n > \frac{1}{0.2} = \frac{1}{\frac{1}{5}} = 5$ $\qquad n > 5$ pomeni

$n \geq 6$



$|a_n| = 0.04$

$n = 5$

Izven okolice $(-0.04, 0.04)$ so: a_1, a_2, a_3, a_4, a_5.

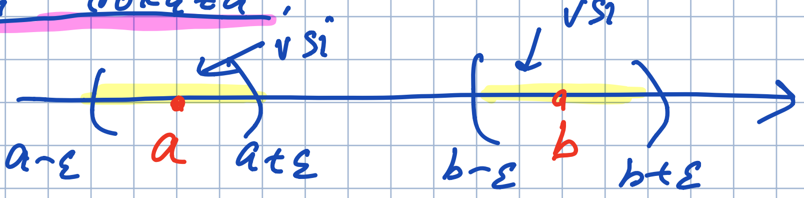
Lastnosti:

① Če zaporedje $(a_n)_{n\in\mathbb{N}}$ ima limito, potem je limita samo ena.

② Če zaporedje $(a_n)_{n\in\mathbb{N}}$ ima limito, potem je omejeno

Skica dokaza:

1)

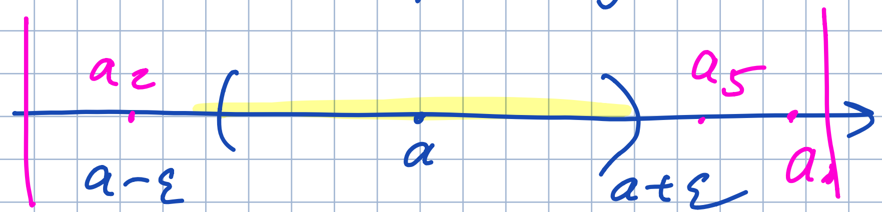


če bi bilo $\lim\limits_{n\to\infty} a_n = a$ in $\lim\limits_{n\to\infty} a_n = b$

$a \neq b$

$\varepsilon = \frac{|b-a|}{4}$ protislovje!

2)

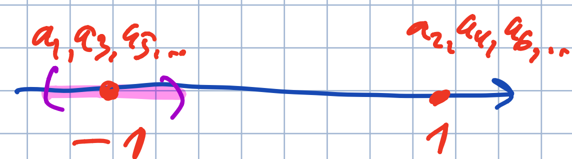


Če zaporedje $(a_n)_{n\in\mathbb{N}}$ ima limito, je konvergentno

če zaporedje $(a_n)_{n\in\mathbb{N}}$ nima limite, je divergentno

Primer 3. $a_n = (-1)^n$

$-1, 1, -1, 1, -1, 1, \ldots$

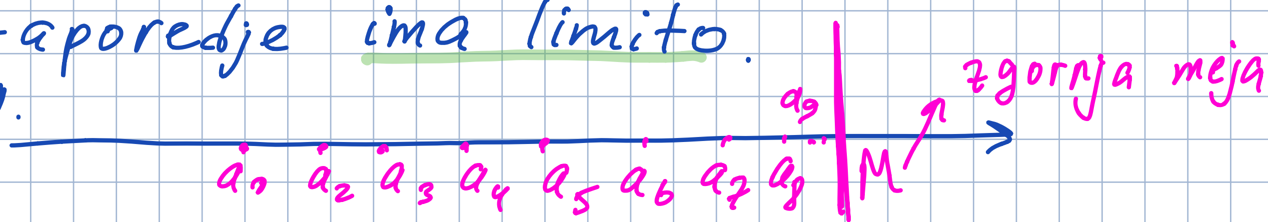


divergentno.

Trditev (zadostni pogoj za konvergenco zaporedja) 1. Vsako naraščajoče in navzgor omejeno zaporedje ima limito.

2. Vsako padajoče in navzdol omejeno zaporedje ima limito.

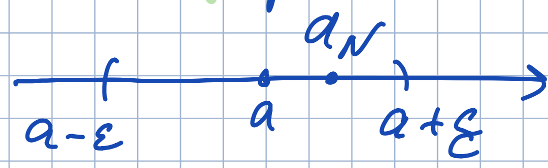
1.



V točki 1: $\lim\limits_{n\to\infty} a_n = \sup\limits_{n\in\mathbb{N}} \{a_n\}$

V točki 2: $\lim\limits_{n\to\infty} a_n = \inf\limits_{n\in\mathbb{N}} \{a_n\}$.

Opomba ε od besede "error" - napaka



$a_N \approx a$

ε majhno

Podzaporedja in stekališča

$(a_n)_{n\in\mathbb{N}}$ zaporedje

$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6} \Rightarrow 1, \frac{1}{3}, \frac{1}{5}, \dots$.

podzaporedje

Vzamemo $n_1 < n_2 < \dots$

$(a_{n_k})_{k\in\mathbb{N}}$ - podzaporedje zaporedja $(a_n)_{n\in\mathbb{N}}$

Definicija Stekališče zaporedja $(a_n)_{n\in\mathbb{N}}$ je limita nekega podzaporedja.

Primer. $a_n = (-1)^n$

$-1, 1, -1, 1, -1, 1, \dots$.

$(a_{2n-1})_{n\in\mathbb{N}}$ podzaporedje: $-1, -1, -1, \dots$

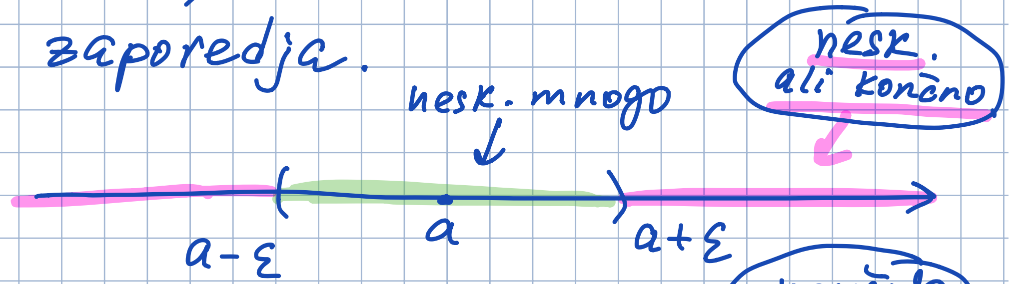
$(a_{2n})_{n\in\mathbb{N}}$ podzaporedje: $1, 1, 1, \dots$

2 stekališči: -1 in 1

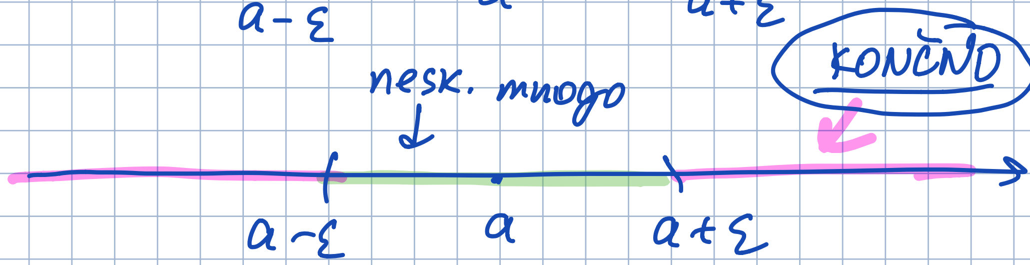
$-1 = \lim\limits_{n\to\infty} a_{2n-1}$ in $1 = \lim\limits_{n\to\infty} a_{2n}$

Trditev: Število a je stekališče zaporedja $(a_n)_{n\in\mathbb{N}}$, če $\forall \varepsilon>0$ v okolici $(a-\varepsilon, a+\varepsilon)$ leži neskončno mnogo členov zaporedja.

Stekališče:



Limita:



Opomba 1. Stekališč je lahko več, limita samo ena.

2. Če zaporedje je konvergentno, je limita edino stekališče.

Izrek (Bolzano - Weirstrass) Vsako omejeno zaporedje ima vsaj eno stekališče (= vsaj eno konvergentno podzaporedje)

Računska pravila za limite

$(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ - konvergentni in $\lim\limits_{n\to\infty} a_n = a$, $\lim\limits_{n\to\infty} b_n = b$. Potem:

1. $(a_n \pm b_n)_{n\in\mathbb{N}}$ je konvergentno in
$$\lim_{n\to\infty} (a_n \pm b_n) = a \pm b.$$

2. $(a_n b_n)_{n\in\mathbb{N}}$ je konvergentno in
$$\lim_{n\to\infty} (a_n \cdot b_n) = a\cdot b$$

Posebej: $\lim\limits_{n\to\infty} (\lambda a_n) = \lambda a, \quad \lambda \in \mathbb{R}$

(3) Če je $b_n \neq 0$ za vsak n in $b \neq 0$:
$\left(\frac{a_n}{b_n}\right)_{n\in\mathbb{N}}$ je konvergentno in

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{a}{b}.$$

Pazimo: $a, b \in \mathbb{R}$

$\frac{\infty}{\infty}$

Primer 1. $\lim\limits_{n\to\infty} \frac{\overset{\to\infty}{n+2}}{\underset{\to\infty}{3n-1}} =$

$$= \lim_{n\to\infty} \frac{\cancel{n}\cdot\left(1+\frac{2}{n}\right)}{\cancel{n}\cdot\left(\underset{\downarrow 3}{3} - \underset{\downarrow 0}{\frac{1}{n}}\right)} = \frac{1}{3}$$

$\infty - \infty$

Primer 2 $\lim\limits_{n\to\infty} \left(\underset{\downarrow\infty}{\sqrt{n+1}} - \underset{\downarrow\infty}{\sqrt{n}}\right) =$

$$= \lim_{n\to\infty} \frac{\left(\sqrt{n+1}-\sqrt{n}\right)\cdot\left(\sqrt{n+1}+\sqrt{n}\right)}{\sqrt{n+1}+\sqrt{n}} =$$

$(a-b)(a+b) = a^2-b^2$

$a = \sqrt{n+1}$

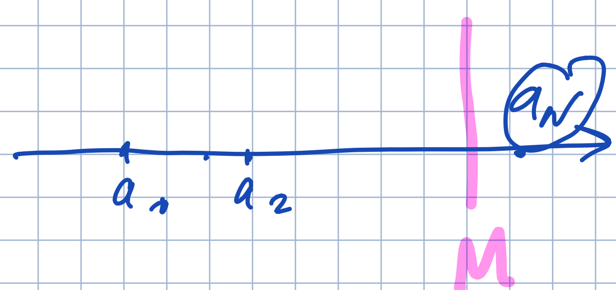
$a^2 = n+1$

$b = \sqrt{n}$

$b^2 = n$

$$= \lim_{n\to\infty} \frac{\cancel{n}+\overset{\to 1}{1}-\cancel{n}}{\underbrace{\underset{\searrow\infty}{\sqrt{n+1}}+\underset{\searrow\infty}{\sqrt{n}}}_{\to\infty}} = 0$$

Def (neskončna limita) $\boxed{\lim\limits_{n\to\infty} a_n = \infty}$:



$\forall M \in \mathbb{R}$ obstaja N:
$a_n \geq M$ za vsak $n \geq N$

Pazi če je $\lim\limits_{n\to\infty} a_n = \infty$, $(a_n)_{n\in\mathbb{N}}$ je divergentno!

Podobno: $\lim_{n\to\infty} a_n = -\infty$ ← divergentno.

Nedoločeni izrazi: $\frac{\infty}{\infty}$, $\frac{0}{0}$, $\infty - \infty$, 1^{∞}

(4) $a_n \le b_n \le c_n$ → za vsak $n \in \mathbb{N}$
če sta $(a_n)_{n\in\mathbb{N}}$ in $(c_n)_{n\in\mathbb{N}}$ konvergentni
z $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = a \in \mathbb{R}$ Potem
je $\lim_{n\to\infty} b_n = a$ (sandvič pravilo).

Primer. $a_n = \frac{\sin n}{n}$. Ali convergira?

$$\underset{\downarrow 0}{-\frac{1}{n}} \le \frac{\sin n}{n} \le \underset{\downarrow 0}{\frac{1}{n}} \quad \text{za vsak } n \in \mathbb{N}$$

$$\Rightarrow \lim_{n\to\infty} \frac{\sin n}{n} = 0.$$

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(5) Če je $\lim_{n\to\infty} a_n = a$ in je f - zvezna funkcija in velja $a_n \in D_f \; \forall n$, $a \in D_f$

Potem je: $\boxed{\lim_{n\to\infty} f(a_n) = f(a)}$

Primer 1

Naj bo $\lim_{n\to\infty} a_n = 0 \Rightarrow \lim_{n\to\infty} \cos(\underset{\downarrow 0}{a_n}) = \cos 0 = 1$

Primer 2

$$\lim_{n\to\infty} \sqrt[n]{5} = \lim_{n\to\infty} 5^{\frac{1}{n} \to 0} = 5^0 = 1$$

Primer $a_1 = 3$, $a_{n+1} = \dfrac{7+3a_n}{3+a_n}$, $n \geq 1$

Utemeljimo konvergenco in izračunajmo limito.

Kandidatke za limito: če limita obstaja, označimo jo z a:

$$\underbrace{a_{n+1}}_{\to a} = \frac{7+3\overbrace{a_n}^{\to a}}{3+\underbrace{a_n}_{\to a}} \Rightarrow \text{vzamemo limito}$$

$$a = \frac{7+3a}{3+a}, \quad a \neq -3 \quad \cdot (3+a)$$

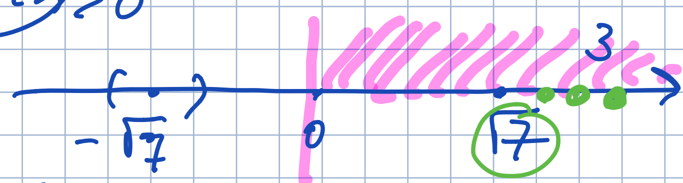
$$a(3+a) = 7+3a \qquad \cancel{3a} + a^2 = 7 + \cancel{3a} \qquad a = \pm\sqrt{7}$$

$a_1 = \sqrt{7}$, $a_2 = -\sqrt{7}$.

$a_n \geq 0 \quad \forall n$. Dokaz. $a_1 \geq 0$ ✓

IP $a_n \geq 0$. $\quad a_{n+1} = \dfrac{\overbrace{7+3a_n}^{>0}}{\underbrace{3+a_n}_{>0}} > 0$ ✓

$-\sqrt{7}$ ne more biti limita!



(a_n) je navzdol omejeno z 0.

Preverimo, da je $a_{n+1} \leq a_n \ \forall n$.

$\underline{a_2 \leq a_1}$ $\quad a_2 = \dfrac{7+3\cdot 3}{3+3} = \dfrac{16}{6} = \dfrac{8}{3}$ $\quad \dfrac{8}{3} < 3$ ✓

$$a_{n+1} = \frac{7+3a_n}{3+a_n} = \frac{9+3a_n-2}{3+a_n} = 3 - \frac{2}{3+a_n}$$

$3\cdot(3+a_n)$

IP: $a_{n+1} \leq a_n$

Preverimo $a_{n+2} \leq a_{n+1}$

$$\frac{9+3a_n}{3+a_n} - \frac{2}{3+a_n} \qquad \frac{A+B}{C} = \frac{A}{C} + \frac{B}{C}$$

$= 3$

$$\underbrace{\cancel{3} - \frac{2}{3+a_{n+1}}}_{= a_{n+2}} \overset{?}{\leq} \underbrace{\cancel{3} - \frac{2}{3+a_n}}_{= a_{n+1}} \quad :(-2)$$

Poljubno padajoče in navzdol omejeno zaporedje ima limito

$$\frac{1}{3+a_{n+1}} \geq \frac{1}{3+a_n} \quad \cdot \underbrace{(3+a_n)\cdot(3+a_{n+1})}_{>0}$$

$$\cancel{3} + a_n \geq \cancel{3} + a_{n+1}$$

$$a_{n+1} \leq a_n \quad \checkmark \quad (\text{po IP})$$

$$\Rightarrow a_{n+2} \leq a_{n+1} \text{ velja!} \Rightarrow \lim_{n\to\infty} a_n = \sqrt{7}$$

Posebne limite

1. $$\lim_{n\to\infty} a^n = \begin{cases} \infty, & a > 1 \\ 1, & a = 1 \\ 0, & 0 < a < 1 \end{cases}$$

npr. $\lim_{n\to\infty} 2^n = \infty$, $\lim_{n\to\infty} \left(\frac{4}{5}\right)^n = 0$

2. $$\lim_{n\to\infty} \frac{n^k}{a^n} = 0$$ npr. $\lim_{n\to\infty} \frac{n^2}{3^n} \overset{\frac{\infty}{\infty}}{=} 0$

$k \in \mathbb{N}$, $a > 1$

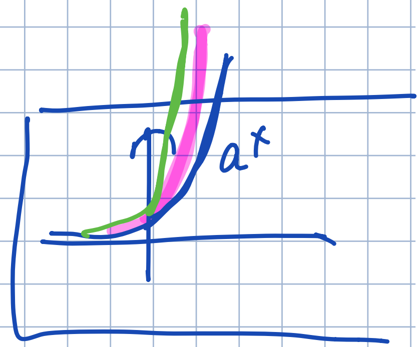
npr $$\lim_{n\to\infty} \frac{n^2+2^n}{n+3^{2n}} \overset{\frac{\infty}{\infty}}{=} \lim_{n\to\infty} \frac{n^2+2^n}{n+9^n} = \quad :9^n$$

$$= \lim_{n\to\infty} \frac{\overbrace{\frac{n^2}{9^n}}^{\to 0} + \overbrace{\left(\frac{2}{9}\right)^n}^{\to 0}}{\underbrace{\frac{n}{9^n}}_{\to 0} + 1} = \frac{0}{1} = 0.$$

3. $$\lim_{n\to\infty} \sqrt[n]{n} = 1$$ $\sqrt[n]{n} = n^{\frac{1}{n}}$, $n \to \infty$, $\frac{1}{n} \to 0$ nedol. izraz

npr. $\lim\limits_{n\to\infty} \sqrt[n]{5n} = \lim\limits_{n\to\infty} \sqrt[n]{5}\cdot \underbrace{\sqrt[n]{n}}_{\downarrow 1} =$

$= \lim\limits_{n\to\infty} 5^{\frac{1}{n}\to 0} = 1$



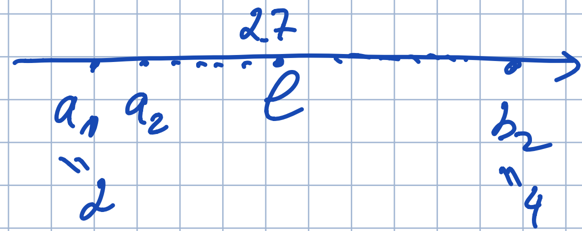
(4) $\lim\limits_{n\to\infty} \underbrace{\left(1+\frac{1}{n}\right)^{n}}_{=a_n} = e$

$\lim\limits_{n\to\infty} \underbrace{\left(1-\frac{1}{n}\right)^{-n}}_{=b_n} = e$

$n \geq 2$

$a_n \uparrow e \approx 2.71828$

$b_n \downarrow e$



$b_2 = \left(1-\frac{1}{2}\right)^{-2} = \left(\frac{1}{2}\right)^{-2} =$

$= \left(\left(\frac{1}{2}\right)^2\right)^{-1} = \left(\frac{1}{4}\right)^{-1} = 4$

21.12.2020 v četrtek 24.12 - petkov urnik.

Primer 1) $\lim\limits_{n\to\infty} \left(1+\frac{1}{n}\right)^{2020}$ 2) $\lim\limits_{n\to\infty} \left(2+\frac{1}{n}\right)^{n}$

$\boxed{\lim\limits_{n\to\infty} \left(1+\frac{1}{n}\right)^{n} = e}$

3) $\lim\limits_{n\to\infty} \left(1+\frac{2}{n}\right)^{n}$

1) $\lim\limits_{n\to\infty} \underbrace{\left(1+\frac{1}{n}\right)}_{\downarrow 1}^{2020} = 1$ (ni nedoločenosti!)

2) $\lim\limits_{n\to\infty} \underbrace{\left(2+\frac{1}{n}\right)}_{\downarrow 2}^{n\to\infty} = \infty$ (ker je $\lim\limits_{n\to\infty} 2^n = \infty$)

$\frac{1}{n} \to 0$

3) $\lim\limits_{n\to\infty} \underbrace{\left(1+\frac{2}{n}\right)}_{\downarrow 1}^{n\to\infty} \quad 1^{\infty} \quad = \lim\limits_{n\to\infty} \left(1+\frac{1}{\frac{n}{2}}\right)^{\frac{n}{2}\cdot 2}$ $\to e$

$\frac{a}{b} = \frac{1}{\frac{b}{a}}$

$k = \frac{n}{2}$ $= \lim\limits_{k\to\infty} \underbrace{\left(1+\frac{1}{k}\right)^{k}}_{=e}{}^{\cdot 2} = e^2$

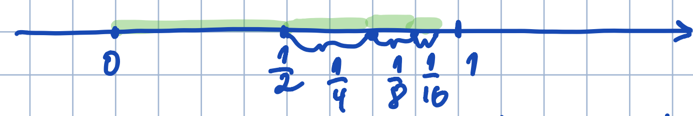
Številske vrste

Def. številska vrsta je izraz $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots$

$a_n \in \mathbb{R} \ \forall n.$

npr. $\sum_{n=1}^{\infty} n = 1+2+3+4+\dots$, $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$

$\underbrace{\qquad}_{\infty \to \text{divergira}}$ $\qquad = 1$



Def. Delna vsota:

$S_k = \sum_{n=1}^{k} a_n = a_1 + a_2 + \dots + a_k$ $\qquad S_4 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}$

k. delna vsota, oznaka $\quad a_n$ - splošni člen vrste

Def. Vrsta $\sum_{n=1}^{\infty} a_n$ je konvergentna, če obstaja

$\lim_{k\to\infty} S_k = s.$ $\quad$ Število s se imenuje vsota vrste.

Sicer, če $\lim_{k\to\infty} S_k$ ne obstaja, vrsta divergira (vrste ne moremo sešteti)

Trditev (potreben pogoj za konvergenco vrste). Če vrsta

$\sum_{n=1}^{\infty} a_n$ konvergira $\Rightarrow \lim_{n\to\infty} a_n = 0$

Dokaz $\quad \sum_{n=1}^{\infty} a_n = s$

$a_n = S_n - S_{n-1}$ $\quad$ vzamemo limito: $\quad \lim_{n\to\infty} a_n = \underbrace{\lim_{n\to\infty} S_n}_{\to s} - \underbrace{\lim_{n\to\infty} S_{n-1}}_{\to s}$

$\Rightarrow \lim_{n\to\infty} a_n = 0$

$\lrcorner$

Posledica če $\lim_{n\to\infty} a_n \neq 0 \Rightarrow$ vrsta divergira.

Konvergira $\equiv$ je konvergentna

Primer $\sum_{n=1}^{\infty} \frac{2n}{n+1}$ $\qquad a_n = \frac{2n}{n+1}$ $\qquad \lim_{n\to\infty} \frac{2n}{n+1} = \lim_{n\to\infty} \frac{\cancel{n}\cdot 2}{\cancel{n}(\underbrace{1+\frac{1}{n}}_{\to 1})}$

$= 2 \Rightarrow$ vrsta divergira

$\sum_{n=1}^{\infty} n^2 \qquad \lim_{n\to\infty} a_n = \lim_{n\to\infty} n^2 = \infty \neq 0 \Rightarrow$ divergira!

Kaj pa, če $\lim\limits_{n\to\infty} a_n = 0$ ↗ konvergira ↘ divergira.

Primer 1 $\sum\limits_{n=1}^{\infty}\left(\frac{1}{2}\right)^n$ $a_n = \left(\frac{1}{2}\right)^n = \frac{1}{2^n}$ $\lim\limits_{n\to\infty}\frac{1}{2^n} = 0$

Konvergira:

$$S_k = \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \dots + \left(\frac{1}{2}\right)^k = \frac{\cancel{\frac{1}{2}}\cdot\left(1-\left(\frac{1}{2}\right)^{k-1}\right)}{\underbrace{1-\frac{1}{2}}_{=\cancel{1/2}}} =$$

$$= 1 - \underbrace{\left(\frac{1}{2}\right)^{k-1}}_{\to 0} \qquad \lim\limits_{k\to\infty} S_k = 1$$

Primer 2 $\sum\limits_{n=1}^{\infty}\frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$ harmonična vrsta

$a_n = \frac{1}{n}$ $\lim\limits_{n\to\infty}\frac{1}{n} = 0$ Ali vrsta konvergira?

Vrsta je divergentna: $\lim\limits_{k\to\infty} S_k = \infty$

$$S_{2^{k+1}} = 1 + \frac{1}{2} + \overbrace{\left(\underbrace{\frac{1}{3}}_{\geq\frac{1}{4}} + \underbrace{\frac{1}{4}}_{\geq\frac{1}{4}}\right)}^{\geq 1/2} + \overbrace{\left(\underbrace{\frac{1}{5}}_{\geq\frac{1}{8}} + \underbrace{\frac{1}{6}}_{\geq\frac{1}{8}} + \underbrace{\frac{1}{7}}_{\geq\frac{1}{8}} + \underbrace{\frac{1}{8}}_{\geq\frac{1}{8}}\right)}^{\geq 1/2} + \dots \underbrace{\left(\frac{1}{2^k+1} + \dots + \frac{1}{2^{k+1}}\right)}_{\geq\frac{1}{2}} \geq 1 + (k+1)\cdot\frac{1}{2}$$

$\to \infty,\ k\to\infty$

$k=3$ $S_{2^3} \geq 1 + \frac{4}{2} = \frac{5}{2}$ torej $S_8 \geq \frac{5}{2} = 2.5$

$k=4$ $S_{16} \geq 1 + \frac{5}{2} = \frac{7}{2} = 3.5$

$1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{16} > 3.5$, $1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{32} > 4.5$

$\lim\limits_{k\to\infty} S_k = \infty \Rightarrow$ harmonična vrsta divergira

Geometrijska vrsta

$\sum\limits_{n=0}^{\infty} aq^n$, $a \neq 0$. $\sum\limits_{n=0}^{\infty} a\cdot q^n = a + a\cdot q + a\cdot q^2 + \dots$

$$S_k = \sum_{n=0}^{k} a\cdot q^n = a + aq + \dots + aq^k = a\cdot(1 + q + \dots + q^k)$$

$$1 + q + \dots + q^k = \frac{1-q^{k+1}}{1-q} \quad \text{ker:}$$

$(1+q+q^2)(1-q) = 1-q^3$

$1 + \cancel{q} + \cancel{q^2} - \cancel{q} - \cancel{q^2} - q^3$

$$(1+q+\dots+q^k)(1-q) = 1-q^{k+1}$$

$$1 + \cancel{q} + \cancel{q^2} \dots + \cancel{q^k} - \cancel{q} - \cancel{q^2} - \dots \cancel{q^k} - q^{k+1} = 1 - q^{k+1}.$$

$\Rightarrow S_k = a \cdot (1 + q + \ldots + q^k) = a \cdot \frac{1 - q^{k+1}}{1-q}$, $q^{k+1} \to 0$, če $|q| < 1$

Trditev. Geom. vrsta $\sum_{n=0}^{\infty} a q^n$ ($a \neq 0$) konvergira natanko tedaj, ko je $|q| < 1$. V tem primeru je

$$\boxed{\sum_{n=0}^{\infty} a q^n = \frac{a}{1-q}}$$

$$\sum_{n=0}^{\infty} a q^n = \lim_{k \to \infty} S_k = \lim_{k \to \infty} a \frac{1 - q^{k+1}}{1-q} = a \cdot \frac{1}{1-q}$$

($q^{k+1} \to 0$)

Če je $|q| \geq 1$, geom. vrsta $\sum_{n=0}^{\infty} a q^n$ divergira.

Primer 1 $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{2} \cdot \frac{1}{1 - \frac{1}{2}} = 1$

$a = 1$, $q = \frac{1}{2}$. Ker je $|\frac{1}{2}| < 1 \Rightarrow$ konvergira

Primer 2 $\sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n = \frac{1}{1 - (-\frac{1}{2})} = \frac{1}{\frac{3}{2}} = \frac{2}{3}$

$a = 1$, $q = -\frac{1}{2}$ $|-\frac{1}{2}| < 1 \Rightarrow$ konvergira

Primer 3 $\sum_{n=0}^{\infty} \left(\frac{3}{2}\right)^n$ geom. divergira $q = \frac{3}{2} > 1$.

II. način $a_n = \left(\frac{3}{2}\right)^n$ $\lim_{n \to \infty} \left(\frac{3}{2}\right)^n = \infty \neq 0 \Rightarrow$ divergira.

Vrste s pozitivnimi členi

Predpostavimo: $a_n > 0$.

Trditev (primerjalni test) $\sum_{n=1}^{\infty} a_n$ in $\sum_{n=1}^{\infty} b_n$

predpostavimo $0 \leq a_n \leq b_n \ \forall n$

1. Če $\sum_{n=1}^{\infty} b_n$ konvergira $\Rightarrow \sum_{n=1}^{\infty} a_n$ konvergira.
2. Če $\sum_{n=1}^{\infty} a_n$ divergira $\Rightarrow \sum_{n=1}^{\infty} b_n$ divergira

Primer 1 $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \ldots$

$\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{1}{\sqrt{n}} = 0$ → konv.? → dir.?

$\frac{1}{\sqrt{n}} \geq \frac{1}{n} \iff n \geq \sqrt{n} \quad : \sqrt{n} \qquad \sqrt{n} \geq 1 \quad \forall n \in \mathbb{N}$

npr. $n=4$ $\quad \frac{1}{\sqrt{4}} > \frac{1}{4} \quad \frac{1}{2} > \frac{1}{4}$

$0 \leq \underbrace{\frac{1}{n}}_{a_n} \leq \underbrace{\frac{1}{\sqrt{n}}}_{b_n} \qquad \sum_{n=1}^{\infty} \frac{1}{n}$ divergira $\Rightarrow \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ divergira.

Primer 2 $\quad a_n = \frac{|\sin n|}{2^n} \qquad \sum_{n=1}^{\infty} \frac{|\sin n|}{2^n} \qquad |\sin n| \leq 1 \quad \forall n.$

$0 \leq \frac{|\sin n|}{2^n} \leq \frac{1}{2^n} = \left(\frac{1}{2}\right)^n$

Ker $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$ konvergira $\Rightarrow \sum_{n=1}^{\infty} \frac{|\sin n|}{2^n}$ konvergira.

Alternirajoče vrste

$\sum_{n=1}^{\infty} (-1)^{n+1} \cdot a_n \ , \quad \boxed{a_n \geq 0.} \qquad \left(\text{tudi } \sum_{n=1}^{\infty} (-1)^n a_n\right)$

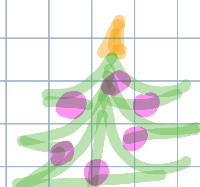
$\sum_{n=1}^{\infty} (-1)^{n+1} a_n = a_1 - a_2 + a_3 - a_4 + a_5 - \ldots$

Trditev (Leibnizov test) če velja:

1. $(a_n)_{n \in \mathbb{N}}$ je padajoče : $a_{n+1} \leq a_n \ \forall n$
2. $\lim_{n \to \infty} a_n = 0$

Potem vrsta $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ konvergira

24. 12. 2020



VESELE PRAZNIKE

Primer $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$

$a_n = \frac{1}{n}$ $\qquad$ Leibnizov test:

1) $\frac{1}{n+1} \leq \frac{1}{n} \quad \cdot n \cdot (n+1) \qquad n \leq n+1$ ✓

2) $\lim_{n \to \infty} \frac{1}{n} = 0 \quad \Rightarrow$ vrsta konvergira

Pazi : $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$ harmonična vrsta, ki divergira.

Opomba. Znano je : $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \ln 2$

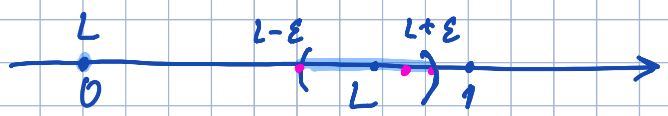
Kvocientni in korenski test : $\sum_{n=1}^{\infty} a_n$

1. Kvocientni test.

Predpostavimo, da obstaja $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| =: L$ $\Rightarrow L \geq 0$

- če je $L < 1$, vrsta konvergira.
- če je $L > 1$, vrsta divergira
- če je $L = 1$, test odpove

Skica dokaza. Naj bo $L < 1$.



Naj bo ε tak, da je

$(L-\varepsilon, L+\varepsilon) \subseteq (0,1)$ $\quad 0 < L+\varepsilon < 1$

Od nekega N dalje velja: $\forall n \geq N$ $\quad q < 1$

$\left|\frac{a_{n+1}}{a_n}\right| < \underbrace{L+\varepsilon}_{q}$ imamo $|a_{n+1}| < q|a_n| < q^2|a_{n-1}| < q^3|a_{n-2}| \ldots$

Primerjava z geom. vrsto, z $q < 1$ in $q > 0$ $\Rightarrow$ vrsta konvergira.

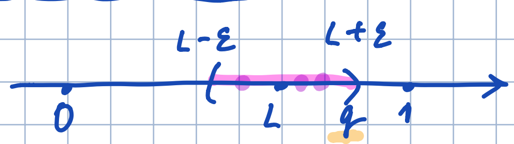
Primer $L > 1$: podobno: DN

2. Korenski test

Predpostavimo, da obstaja $\lim_{n\to\infty} \sqrt[n]{|a_n|} =: L$ (definirajmo)

- če je $L < 1$, vrsta konvergira.
- če je $L > 1$, vrsta divergira
- če je $L = 1$, test odpove

Skica dokaza



ε tak, da je $L < \underbrace{L+\varepsilon}_{=q} < 1$.

Od nekega N dalje velja: (za $n \geq N$)

$\sqrt[n]{|a_n|} < q$, $|a_n| < q^n$ $\quad \sum_{n=1}^{\infty} q^n$ konvergira, ker $0 < q < 1$

$\Rightarrow$ vrsta konvergira.

$L > 1$: podobno, DN.

Primer $\sum_{n=1}^{\infty} \frac{x^n}{n!}$, $x \in \mathbb{R}$.

$n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n$

npr. $2! = 2$, $4! = 1 \cdot 2 \cdot 3 \cdot 4 = 24$

Dogovor: $0! = 1$

Korenski test. $\sqrt[n]{\left|\frac{x^n}{n!}\right|} = \frac{\sqrt[n]{|x|^n}}{\sqrt[n]{n!}} \to$ limita?

Kvocientni test: $x \neq 0$!

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}}\right| = \left|\frac{x^{n+1} \, n!}{(n+1)! \, x^n}\right| = \left|\frac{x}{n+1}\right|$$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x}{n+1}\right| = \underset{L}{0} < 1 \Rightarrow \text{vrsta konvergira!}$$

$x = 0$ $\sum_{n=1}^{\infty} \frac{0^n}{n!} = 0 + 0 + \ldots = 0$.

Vrsta konvergira $\forall x \in \mathbb{R}$

Opomba $\boxed{\sum_{n=0}^{\infty} \frac{x^n}{n!}} = e^x$ ← Taylorjeva vrsta za $f(x) = e^x$.

Če je $x = 1$ $\sum_{n=0}^{\infty} \frac{1}{n!} = e$

Limita in zveznost funkcije

Primer

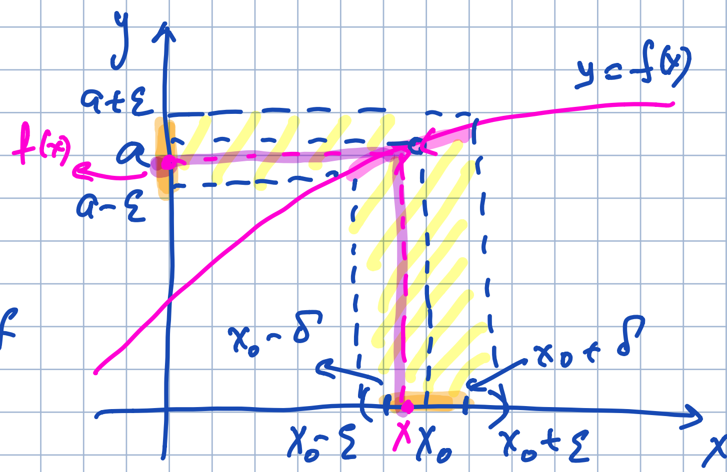
Naj bo $f(x) = \begin{cases} 1, & \text{če je } x \neq 0 \\ 2, & \text{če je } x = 0 \end{cases}$

$$\lim_{\substack{x\to 0 \\ x\neq 0}} f(x) = 1$$

28. 12. 2020

Naj bo $x_0 \in \mathbb{R}$ da velja:

$(x_0 - \varepsilon, x_0 + \varepsilon) \setminus \{x_0\} \subseteq D_f$



Def 1 Število $a \in \mathbb{R}$ je limita funkcije f, ko gre x proti x_0, pišemo $a = \lim_{x\to x_0} f(x)$, če

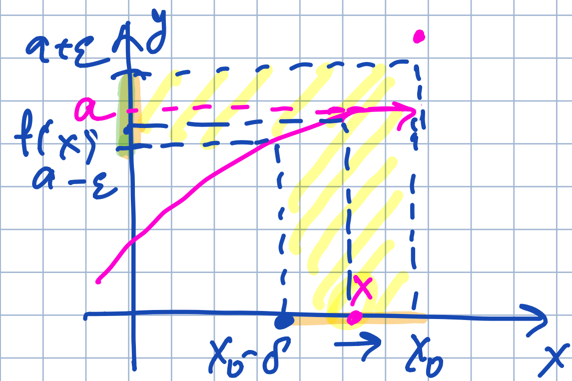
$\forall \varepsilon > 0 \quad \exists \delta > 0$: če $|x - x_0| < \delta$ in $x \neq x_0$

potem je $|f(x) - a| < \varepsilon$.

$\forall$: za vsak
$\exists$: obstaja

Def 2 (enostranski limiti). Naj bo $(x_0, x_0 + \varepsilon) \in D_f$ za nek $\varepsilon > 0$

Število a je leva limita funkcije f v točki x_0, pisava



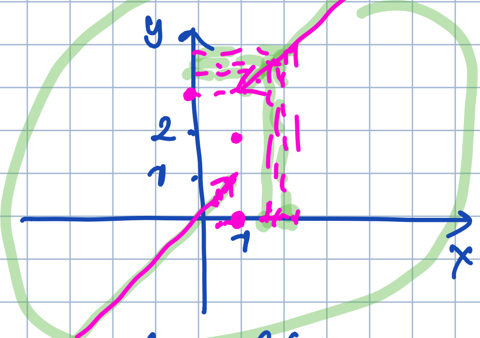
$$a = \lim_{x \uparrow x_0} f(x) = \lim_{x \nearrow x_0} f(x) \text{, če velja:}$$

$\forall \varepsilon > 0 \; \exists \delta > 0$: če je $x_0 - \delta < x < x_0$:
$|f(x) - a| < \varepsilon$.

Podobno (DN): [isto, samo $x_0 < x < x_0 + \delta$]

$\lim_{x \downarrow x_0} f(x) = a \leftarrow$ desna limita

Primer. $f(x) = \begin{cases} x, & x < 1 \\ 2, & x = 1 \\ x+2, & x > 1 \end{cases}$



$\lim_{x \uparrow 1} f(x) = 1$, $\lim_{x \downarrow 1} f(x) = 3$

$\lim_{x \to 1} f(x)$ ne obstaja

Računska pravila

Naj bo $\lim_{x \to x_0} f(x) = a$, $\lim_{x \to x_0} g(x) = b$.

Potem je:

1. $\lim\limits_{x\to x_0} (f(x) \pm g(x)) = a \pm b$
2. $\lim\limits_{x\to x_0} (\lambda \cdot f(x)) = \lambda a, \quad \lambda \in \mathbb{R}.$
3. $\lim\limits_{x\to x_0} (f(x)\cdot g(x)) = a\cdot b$
4. $\lim\limits_{x\to x_0} \frac{f(x)}{g(x)} = \frac{a}{b}, \quad b \neq 0$ in $g(x) \neq 0$ v neki okolici x_0
5. Če je $h(x)$ zvezna, potem je $\lim\limits_{x\to x_0} h(f(x)) = h(a)$

Primer

$$\lim_{x\to 0} \frac{\overset{\to 0}{x^3+2x^2}}{\underset{\to 0}{2x^2-x}} = \lim_{x\to 0}\frac{\overset{\to 0}{x^2+2x}}{\underset{\to -1}{2x-1}} = 0$$

nedoločen izraz $\frac{0}{0}$.

$$\lim_{x\to 3}\frac{x^3+2x^2}{2x^2-x} = \frac{27+18}{18-3} = \frac{45}{15} = 3.$$

$$\lim_{x\to\infty}\frac{\overset{\to\infty}{x^3+2x^2}}{\underset{\to\infty}{2x^2-x}} \overset{:x}{=} \lim_{x\to\infty}\frac{\overset{\to\infty}{x^2+2x}}{\underset{\to\infty}{2x-1}} \overset{:x}{=}$$

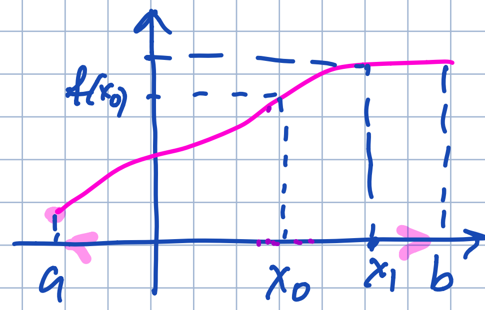
$$= \lim_{x\to\infty}\frac{\overset{\to\infty}{x+2}}{\underset{\to 2}{2-\frac{1}{x}}} = \infty.$$

Zveznost funkcije

Naj bo $(x_0-\varepsilon, x_0+\varepsilon) \subseteq D_f$, tudi $x_0 \in D_f$.

Def. Funkcija f je zvezna v točki x_0, Če obstaja $\lim\limits_{x \to x_0} f(x) = f(x_0)$

f je zvezna na (a, b), če je zvezna v vsaki točki $x_0 \in (a, b)$



(graf je neprekinjena krivulja)

Če je f definirana na $[a, b]$, potem za zveznost zahtevamo zveznost na (a, b) in:

- $\lim\limits_{x \downarrow a} f(x) = f(a)$, (zvezna z desne v a)
- $\lim\limits_{x \uparrow b} f(x) = f(b)$ (zvezna z leve v točki b)

Trditev. Naj bo f definirana na (a, b) in $x_0 \in (a, b)$. Potem je f zvezna v točki x_0 natanko tedaj ko obstajata $\lim\limits_{x \downarrow x_0} f(x)$, $\lim\limits_{x \uparrow x_0} f(x)$ in

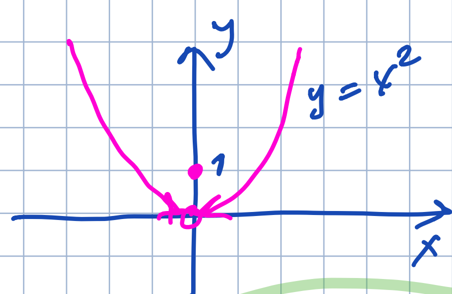
$$\boxed{\lim_{x \downarrow x_0} f(x) = \lim_{x \uparrow x_0} f(x) = f(x_0)}$$

x pada proti x_0

Opomba $\lim\limits_{x \downarrow x_0} f(x) = \lim\limits_{x \uparrow x_0} f(x)$ pomeni, da obstaja $\lim\limits_{x \to x_0} f(x)$.

Primer

$$f(x) = \begin{cases} x^2, & x \neq 0 \\ 1, & x = 0 \end{cases}$$



$$\lim_{x \uparrow 0} f(x) = 0 = \lim_{x \downarrow 0} f(x) \Rightarrow \lim_{x \to 0} f(x) = 0$$

Ali je f zvezna v točki $x_0 = 0$?

NE, ker $f(0) = 1 \neq 0 = \lim_{x \to 0} f(x)$

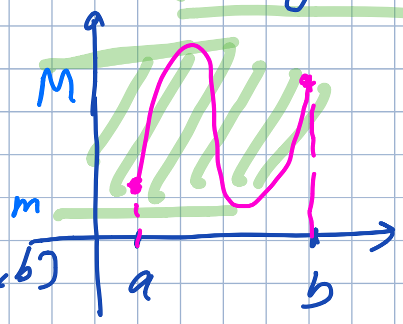
Izreki o zveznih funkcijah

Izrek 1 Naj bo funkcija f zvezna na $[a,b]$. Potem je f omejena na $[a,b]$. → zgornja meja

$\exists\, m, M \in \mathbb{R}:$

$\ni\; m \le f(x) \le M \quad \forall x \in [a,b]$

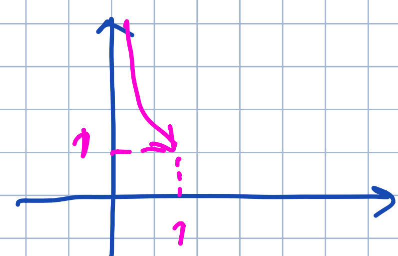
sp. meja



Opomba. Izrek ne velja, če vzamemo (a,b).

npr. $f(x) = \frac{1}{x}, \quad x \in (0,1)$

f je neomejena od zgoraj!



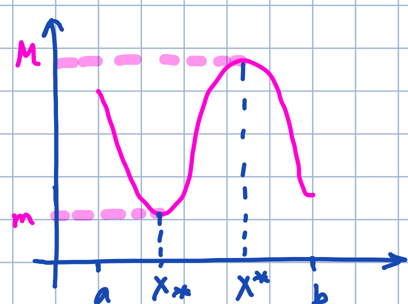
4. 1. 2021

Izrek 2 Naj bo $f: [a,b] \to \mathbb{R}$ zvezna in m in M kot v izreku 1.

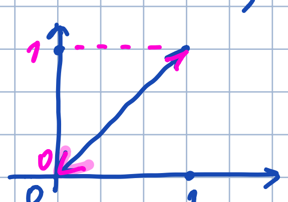
$m = \inf\{f(x) : x \in [a,b]\}, \quad M = \sup\{f(x) : x \in [a,b]\}$

Potem obstajata x_* in $x^* \in [a,b]$:

$f(x_*) = m, \quad f(x^*) = M$

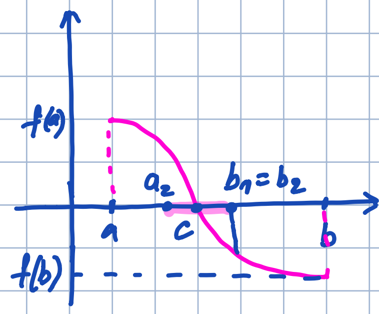


Pazi: $f(x) = x, \quad x \in (0,1)$



$m = 0$
$M = 1$

Ne obstaja $x_* : f(x_*) = 0$!



Izrek 3 Naj bo $f:[a,b]\to\mathbb{R}$ zvezna in $f(a)\cdot f(b)<0$ ($f(a)$ in $f(b)$ sta različnega predznaka). Potem obstaja $c\in(a,b)$: $f(c)=0$.

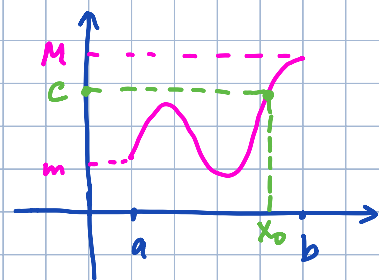
Skica dokaza: (metoda bisekcij)

1) $a=a_0$, $b=b_0$, $[a,b]=[a_0,b_0]$

Naj bo $[a_1,b_1]$ nov interval, kjer je $a_1=a_0$, b_1 pa razpolavlja $[a,b]$ da je $f(a_1)\cdot f(b_1)<0$, ta korak večkrat ponovimo.

Po n korakih $f(a_n)f(b_n)<0$. $c=\lim_{n\to\infty}a_n=\lim_{n\to\infty}b_n$

$f(c)=0$. $\lrcorner$



Izrek 4 $f:[a,b]\to\mathbb{R}$ zvezna

Naj bo $m=\inf\{f(x):x\in[a,b]\}$,

$M=\sup\{f(x):x\in[a,b]\}$

Potem $\forall c\in[m,M]$ obstaja $x_0\in[a,b]$: $f(x_0)=c$

Lastnosti zveznih funkcij.

Naj bosta f in g zvezni na intervalu I.

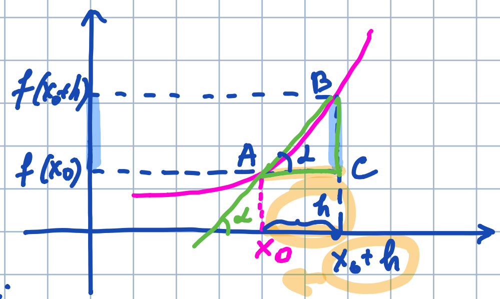
Potem so $f\pm g$, $f\cdot g$, $f\circ g$, $\frac{f}{g}$ (kjer je definirano) zvezni.

Trditev. Vse elementarne funkcije (racion. funkcije, kotne funkcije, arkus funkcije, eksponentna, logaritemska in funkcije dobljene iz njih s pomočjo operacij) so zvezne na svojem definicijskem območju.

Npr. $f(x)=\ln(x^2-5)\cdot\arccos\sqrt{1-x^2}$ je zvezna na D_f.

Odvod funkcije

Kako hitro funkcija raste v točki x_0?



$$\frac{f(x_0+h)-f(x_0)}{h}$$ diferenčni kvocient

Def. Odvod funkcije f v točki x_0 je $\boxed{f'(x_0) = \lim\limits_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}}$, če ta limita obstaja.

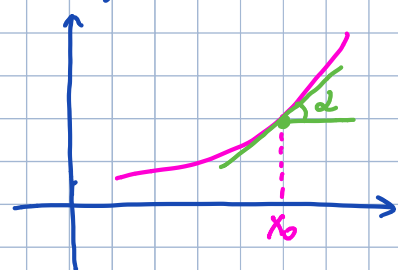
Če limita obstaja, je f odvedljiva v točki x_0.

Geometrijski pomen

$\frac{f(x_0+h)-f(x_0)}{h}$ je enak $\frac{|BC|}{|AC|} = \tan\alpha$

kjer je α kot, ki tvori sekanta AB s pozitivno x-osjo.

$f'(x_0)$ je naklonski koef. tangente na graf funkcije v točki x_0.



$f'(x_0) = \tan\alpha$ (na sliki)

če je enačba tangente $y = kx + n$

potem je $k = f'(x_0)$

Primer 1 Naj bo $f(x) = x^2$ in $x_0 \in \mathbb{R}$. Izračunajmo $f'(x_0)$

$$f'(x_0) = \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h} = \lim_{h\to 0} \frac{(x_0+h)^2 - x_0^2}{h} =$$

$$\lim_{h\to 0} \frac{\cancel{x_0^2} + 2\cdot x_0 h + h^2 - \cancel{x_0^2}}{h} = \lim_{h\to 0} \frac{\cancel{h}\cdot(2x_0+h)}{\cancel{h}} = 2x_0.$$

Pravilo $(x^2)' = 2x$

Primer 2 Zapišemo enačbo tangente na graf $y = x^2$ v točki $x_0 = 3$.

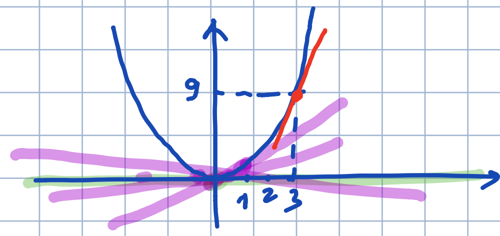
$f(x) = x^2$, $f'(x) = 2x \Rightarrow f'(3) = 6$

iščemo enačbo: $y = kx + n$, $k = f'(3) = 6$

$y = 6x + n$. Tangenta gre skozi točko $(3, 9)$ $(9 = 3^2)$

$9 = 6 \cdot 3 + n \Rightarrow n = 9 - 18 = -9$

R: $y = 6x - 9$



Enačba tangente v točki

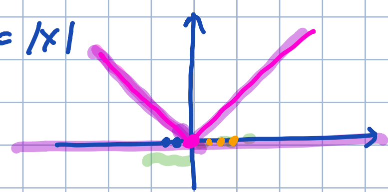
$x_0 = 0$: $y = 0$.

$f'(0) = 2 \cdot 0 = 0$ $y = 0 \cdot x + n$ $y = n$ $(0,0)$: $0 = n \Rightarrow y = 0$

Def. Funkcija f je odvedljiva na I, če je odvedljiva v vsaki točki $x_0 \in I$.

Trditev. Če je f odvedljiva v točki x_0, potem je zvezna v x_0.

Pazi: obrat ne velja! Primer: $f(x) = |x|$



$x_0 = 0$. f je zvezna v točki x_0, ni pa odvedljiva.

Levi odvod: $f'_-(\overset{x_0}{0}) = \lim\limits_{h \uparrow 0} \frac{f(\overset{0}{x_0}+h) - f(x_0)}{h} = \lim\limits_{\substack{h \uparrow 0 \\ h<0}} \frac{|h| - |0|}{h} =$

$= \lim\limits_{h \uparrow 0} \frac{-\cancel{h}}{\cancel{h}} = -1$.

$|h| = \begin{cases} h, & h \geq 0 \\ -h, & h < 0 \end{cases}$

Desni odvod: $f'_+(x_0) = \lim\limits_{h \downarrow 0} \frac{f(x_0+h) - f(x_0)}{h} =$

$= \lim\limits_{\substack{h \downarrow 0 \\ h>0}} \frac{|h| - |0|}{h} = \lim\limits_{h \downarrow 0} \frac{h}{h} = 1$

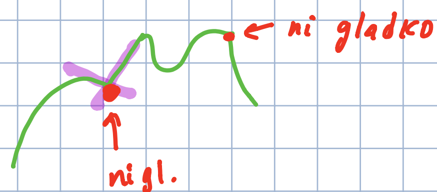
$f'_-(0) \neq f'_+(0)$

$\Rightarrow$ $f'(0)$ ne obstaja

Opomba $f'(x_0)$ obstaja $\Leftrightarrow$ obstajata $f'_-(x_0)$, $f'_+(x_0)$ in $f'_-(x_0) = f'_+(x_0)$

Geometrijsko: v točki $x_0 = 0$ ni tangenta na graf $f(x) = |x|$ ne obstaja! Graf je "oster" v točki 0 (ni gladek)

npr.



Računska pravila za odvajanje.

Naj bosta f in g odvedljivi v točki x_0. Potem:

(1) $f \pm g$ je odvedljiva v točki x_0 in:

$$(f \pm g)(x_0) = f'(x_0) \pm g'(x_0)$$

(2) $f \cdot g$ je odvedljiva v x_0 in:

$$(f \cdot g)'(x_0) = f'(x_0) \cdot g(x_0) + f(x_0) \cdot g'(x_0).$$

(Leibnizovo pravilo).

(3) $\frac{f}{g}$ je odvedljiva v x_0 (če je $g(x_0) \neq 0$) in velja:

$$\left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0) \cdot g(x_0) - f(x_0) \cdot g'(x_0)}{g^2(x_0)}$$

(4) Naj bo g odv. v točki x_0 in f odvedljiva v $g(x_0)$

Potem je $f \circ g$ odvedljiva v x_0 in velja:

$$(f \circ g)'(x_0) = f'(g(x_0)) \cdot g'(x_0)$$ ← verižno pravilo za odvod kompozituma.

Primer Uporabimo $(\sin x)' = \cos x$, $(x^n)' = n \cdot x^{n-1}$. $(x^3)' = 3 \cdot x^2$.

izračunajmo $(\sin(x^3))'$ in $((\sin x)^3)'$

1) $(\sin(x^3))' = \cos(x^3) \cdot 3x^2.$

2) $((\sin x)^3)' = 3 \cdot \sin^2 x \cdot \cos x.$

8. 1. 2021

Tabela odvodov

$$x^{-\frac{1}{2}} = \frac{1}{x^{1/2}} = \frac{1}{\sqrt{x}}$$

(1) $(x^\alpha)' = \alpha \cdot x^{\alpha - 1}$, $\alpha \in \mathbb{R}$.

Posebej: 1) $\alpha = 0$ $x^0 = c$. $c' = 0$

2) $\alpha = 1$ $x' = 1$ 3) $\alpha = \frac{1}{2}$ $x^\alpha = \sqrt{x}$

$$\left(x^{\frac{1}{2}}\right)' = \frac{1}{2} x^{-\frac{1}{2}}, \quad (\sqrt{x})' = \frac{1}{2\sqrt{x}}$$

2) $(\sin x)' = \cos x$

3) $(\cos x)' = -\sin x$

4) $(tg\,x)' = \frac{1}{\cos^2 x}$

5) $(ctg\,x)' = -\frac{1}{\sin^2 x}$

6) $(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$

7) $(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$

8) $(arctg\,x)' = \frac{1}{1+x^2}$

9) $(arcctg\,x)' = -\frac{1}{1+x^2}$

10) $(e^x)' = e^x$

$(a^x)' = ((e^{\ln a})^x)' = (e^{\ln a \cdot x})' = e^{\ln a \cdot x} \cdot \ln a$

$= a^x \cdot \ln a$. Torej: $(a^x)' = a^x \ln a$ $(a>0)$

11) $(\ln x)' = \frac{1}{x}$

$(\log_a x)' \underset{c=e}{=} \left(\frac{\ln x}{\ln a}\right)' = \frac{1}{x \ln a}$

$\log_a b = \frac{\log_c b}{\log_c a}$

$\log_e x = \ln x$

$(\log_a x)' = \left(\frac{\log_e x}{\log_e a}\right)' = \left(\frac{\ln x}{\ln a}\right)' =$

$= \frac{1}{\ln a} \cdot (\ln x)' = \frac{1}{\ln a \cdot x}$

Primer Odvajamo:

1) $(arctg(e^x))' = \frac{1}{1+\underbrace{(e^x)^2}_{e^{2x}}} \cdot e^x = \frac{e^x}{1+e^{2x}}$

$x \to \underline{e^x} \to$
$\to arctg(e^x)$

2) $(e^{\sqrt{x}})' = e^{\sqrt{x}} \cdot (\sqrt{x})' =$

$= e^{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}} = \frac{e^{\sqrt{x}}}{2\sqrt{x}}$.

$$\left(e^{\sqrt{x}}\right)' = e^{\sqrt{x}} \cdot \left(\sqrt{x}\right)' = e^{\sqrt{x}} \cdot \left(x^{\frac{1}{2}}\right)' =$$

$$= e^{\sqrt{x}} \cdot \frac{1}{2} x^{\frac{1}{2}-1} = \frac{1}{2} \cdot e^{\sqrt{x}} \cdot x^{-\frac{1}{2}}.$$

3) $$\left(\frac{x+2}{\sqrt{x}}\right)' = \frac{1 \cdot \sqrt{x} - (x+2)\frac{1}{2\sqrt{x}}}{x}$$

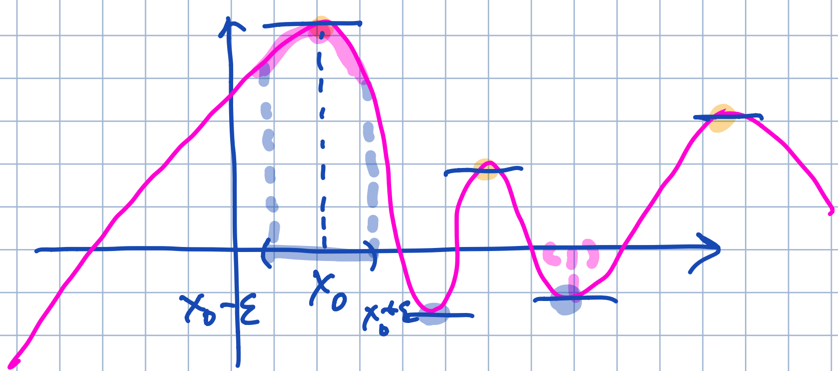
$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

$$\boxed{(\ln x)' = \frac{1}{x}}$$

4) $$\left(\ln\left(\frac{x+2}{\sqrt{x}}\right)\right)' = \frac{1}{\frac{x+2}{\sqrt{x}}} \cdot \left(\frac{x+2}{\sqrt{x}}\right)' =$$

$$= \frac{\sqrt{x}}{x+2} \cdot \frac{\sqrt{x} - (x+2)\frac{1}{2\sqrt{x}}}{x}.$$

Ekstremi funkcij



Def. Funkcija $f(x)$ ima v točki x_0 lokalni maksimum, če: obstaja $\varepsilon > 0$ da velja

$$f(x_0) \geq f(x) \quad \forall x \in (x_0-\varepsilon, x_0+\varepsilon)$$

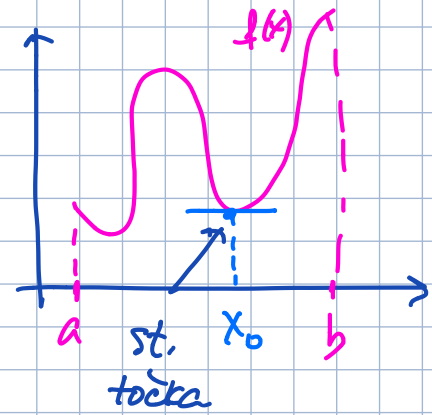
Lok. min: podobno: DN

11. 1. 2021.

Trditev Naj bo $f : [a,b] \to \mathbb{R}$ in naj odvedljiva na $[a,b]$

bo $x_0 \in (a,b)$ lokalni ekstrem (maksimum ali min.)

Potem je: $f'(x_0) = 0$



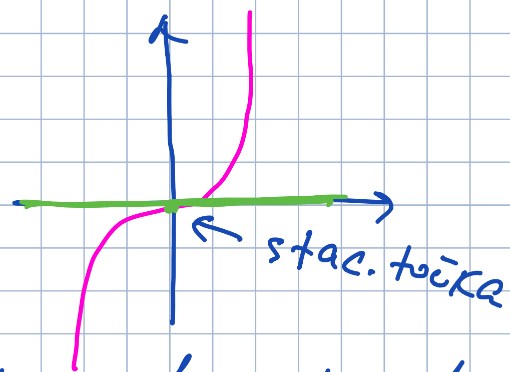
Geometrijsko: tangenta na graf v točki x_0 je vzporedna x-osi.

Opomba: Obrat te trditve ne drži.

npr. $f(x) = x^3$

$f'(x) = 3x^2$

$f'(0) = 0$, ampak $x_0 = 0$ ni lokalni ekstrem!



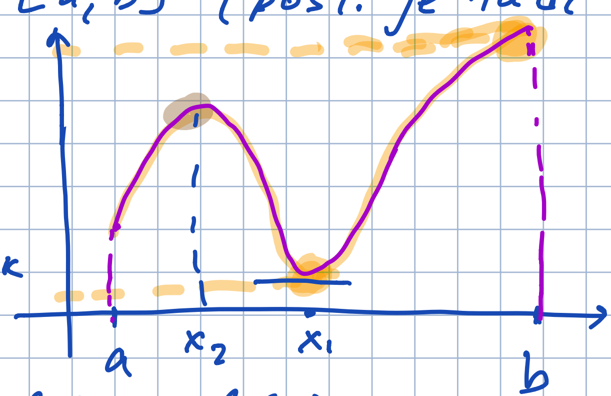
Def. x_0 je stacionarna točka funkcije f, če je $f'(x_0) = 0$

Opomba: stacionarne točke so "kandidatke" za ekstrem.

Iskanje največje in najmanjše vrednosti funkcije f na $[a,b]$

Naj bo f odvedljiva na $[a,b]$ (posl. je tudi zvezna na $[a,b]$).

1) Stac. točke v intervalu (a,b): $x_1, x_2 \dots x_k$



2) izračunajmo:

$f(x_1), f(x_2) \dots, f(x_k), f(a), f(b)$ in izberemo najv. in najmanjšo.

Primer $f(x) = \frac{x}{1+x^2}$. Poiščemo globalne ekstreme na $[1/2, 3]$

$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$

1) $f'(x) = \frac{1\cdot(1+x^2) - x\cdot 2x}{(1+x^2)^2} =$

$= \frac{1+x^2-2x^2}{\underbrace{(1+x^2)^2}_{>0}} = 0$

$1 - x^2 = 0$

$x^2 = 1$

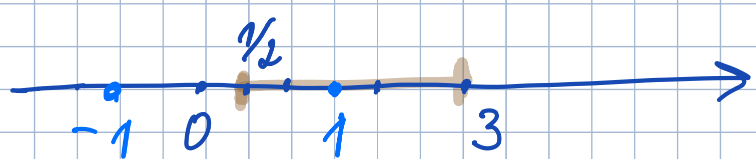
$(x-1)(x+1) = 0$

ni v intervalu

$\cancel{x_1 = -1}$

$x_2 = 1$

stacionarni točki



$x_2 = 1$

2) $f(1)$, $f\left(\frac{1}{2}\right)$, $f(3)$ in jih primerjamo

$f(x) = \frac{x}{1+x^2}$

$f(1) = \frac{1}{1+1} = \frac{1}{2}$, $f\left(\frac{1}{2}\right) = \frac{\frac{1}{2}}{1+\left(\frac{1}{2}\right)^2} =$

$= \frac{1}{2}\,\frac{1}{1+\frac{1}{4}} = \frac{1}{2}\cdot\frac{1}{\frac{5}{4}} = \frac{1}{\cancel{2}}\cdot\frac{\cancel{4}\,2}{5} = \frac{2}{5}$.

$f(3) = \frac{3}{1+3^2} = \frac{3}{10}$

$\frac{1}{2} > \frac{2}{5} \iff 5 > 4$ ✓ $\quad \frac{1}{2} > \frac{3}{10} \iff 10 > 6$ ✓

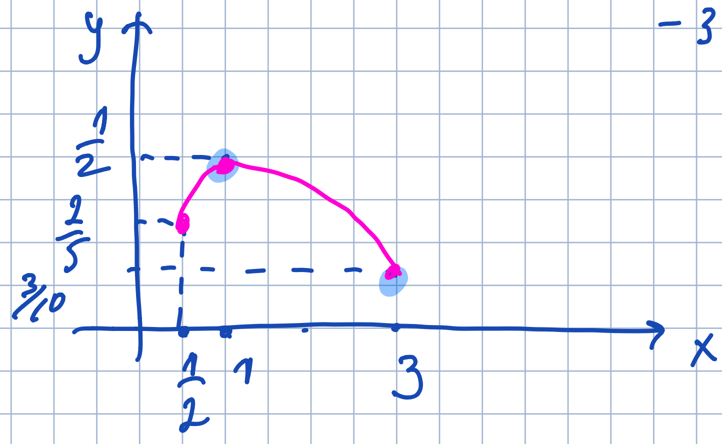
$\frac{3}{10} < \frac{2}{5} \iff 15 < 20$ ✓

Najv. vrednost : $f(1) = \frac{1}{2}$.

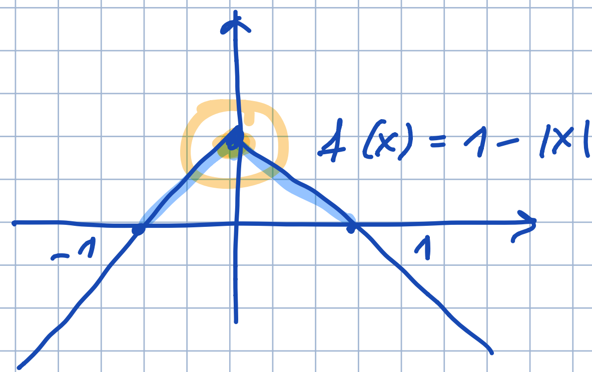
Najm. vrednost $f(3) = \frac{3}{10}$.

Opomba:

pazi: f mora biti odvedljiva na $[a,b]$



Primer $f(x) = 1 - |x|$, $x \in [-1, 1]$



$$f(x) = \begin{cases} 1 + x, & x < 0 \\ 1 - x, & x \geq 0 \end{cases}$$

$$f'(x) = \begin{cases} 1, & x < 0 \\ -1, & x > 0 \end{cases}$$

$f'(x)$ ne obstaja za $x = 0$.

Stac. točk ni! $x = 0$ je glob. maksimum

Intervali naraščanja in padanja

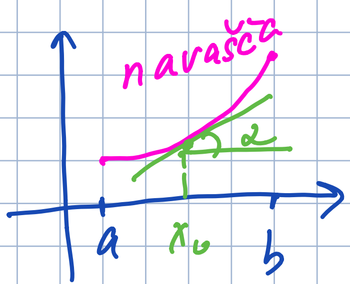
(= intervali monotonosti)

Trditev Naj bo f odvedljiva na (a,b).

1) če je $f'(x) > 0 \ \forall x \in (a,b)$, potem f narašča na (a,b)

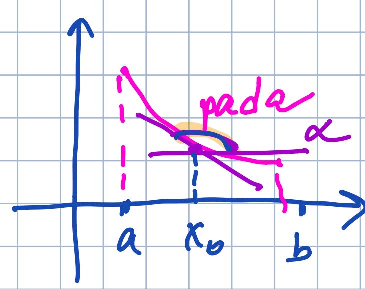
2) če je $f'(x) < 0 \ \forall x \in (a,b)$, potem f pada na (a,b).

$\alpha \in (0, \frac{\pi}{2})$



$f'(x_0) = \underline{tg\, \alpha > 0}$

$\alpha \in (\frac{\pi}{2}, \pi)$



$f'(x_0) = tg\, \alpha < 0$

Ponovitev: matrične enačbe

1. $2A - X = B - 2X$

$-X + 2X = B - 2A$ $\qquad \boxed{X = B - 2A}$

2. $X + AX = B$, A in B 3x3 matriki

$I \cdot X + AX = B$ $\quad I = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$

$\Downarrow$ distr.

$(I + A) \cdot X = B$

$ax + bx = (a+b) \cdot x$

$\to (I + A)^{-1}$

$(i + a) \cdot x = b$

$\underbrace{(I+A)^{-1} \cdot (I+A)}_{= I} \cdot X = (I+A)^{-1} B$

$x = (i+a)^{-1} \cdot b$

$X = (I + A)^{-1} \cdot B$, če je $I + A$ obrnljiva

$[I + A \,\vdots\, B] \rightsquigarrow [I \,\vdots\, (I+A)^{-1} \cdot B]$

$X = BA^{-1} \Rightarrow X^T = (BA^{-1})^T =$

$= (A^{-1})^T B^T = (A^T)^{-1} \cdot B^T$

$[A^T \,\vdots\, B^T] \rightsquigarrow [I \,\vdots\, X^T] \Rightarrow X.$

$[A \,\vdots\, I] \rightsquigarrow [I \,\vdots\, A^{-1}]$

Povzetek:

$XA = B$ $\qquad X = BA^{-1}$ (1)

$AX = B$ $\qquad X = A^{-1}B$ (2)

(2) $[A \mid B] \rightsquigarrow [I \mid A^{-1}B]$

Primer $f(x) = \frac{x}{1+x^2}$. Poiščemo lokalne ekstreme in intervale monotonosti.

$$f'(x) = \frac{1\cdot(1+x^2)-x\cdot 2x}{(1+x^2)^2} = \frac{1-x^2}{(1+x^2)^2}$$

Stac. točke: $f'(x)=0$

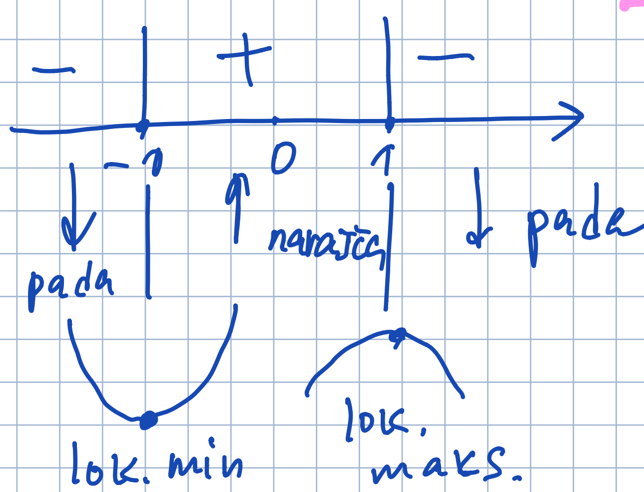
$x_1=-1,\quad x_2=1$

Kdaj f narašča? $f'(x)>0$ $\quad \frac{1-x^2}{(1+x^2)^2}>0$, $\quad (1+x^2)^2 > 0$

$1-x^2>0$

$(1-x)(1+x)>0$

$x=0$ $\quad (1-0)(1+0)>0$

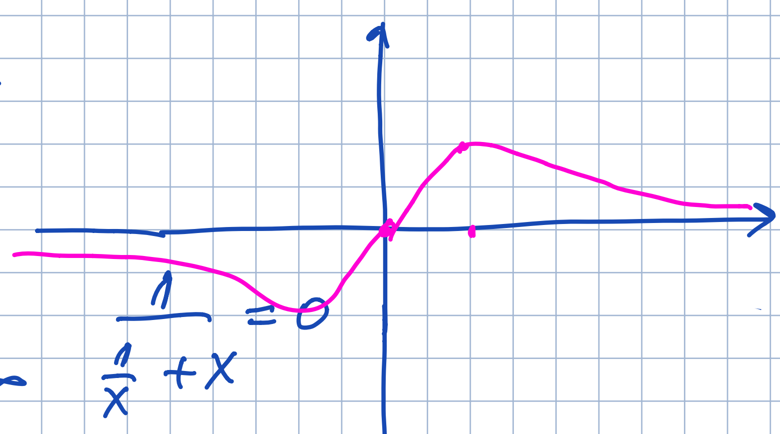


R: $x_1=-1$ lok. min

$x_2=1$ - lok. maks.

$(-\infty,-1)\cup(1,\infty)$ pada, $(-1,1)$ narašča.

Graf: $f(x)=\frac{x}{1+x^2}$



$f(0)=0$

$$\lim_{x\to\infty}\frac{x}{1+x^2} = \lim_{x\to\infty}\frac{1}{\frac{1}{x}+x}=0$$

f - liha $\Rightarrow$ graf je simetričen

L'Hospitalovo pravilo

Trditev. Naj bosta $f(x)$ in $g(x)$ odvedljivi v okolici točke x_0 in je

$$\lim_{x\to x_0} f(x) = \lim_{x\to x_0} g(x) = 0$$

$$\frac{f(x)}{g(x)} \quad \frac{0}{0}$$

Če obstaja limita

$$\lim_{x\to x_0} \frac{f'(x)}{g'(x)} =: L,$$

potem obstaja

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = L.$$

($\approx$ 1698!)

Trditev velja tudi za limite v neskončnosti (t. i. $x_0=\infty$), in za odpravljanje nedoločenosti $\frac{\infty}{\infty}$, tudi za enostranske limite.

L'Hospitalovo pravilo: $\frac{0}{0}$ in $\frac{\infty}{\infty}$.

Primer 1)

$$\lim_{x\to 0} \frac{e^x-1}{x} = \frac{e^0-1}{0} = \frac{0}{0} \overset{L'H}{=} \lim_{x\to 0} \frac{e^x}{1} = \frac{e^0}{1} = 1$$

2)

$$\lim_{x\to 2} \frac{\sqrt{7+x}-3}{x-2} \overset{L'H}{=} \frac{\sqrt{7+2}-3}{2-2} = \frac{0}{0}$$

$$= \lim_{x\to 2} \frac{\frac{1}{2\sqrt{7+x}}}{1} = \lim_{x\to 2} \frac{1}{2\sqrt{7+x}} = \frac{1}{2\cdot 3} = \frac{1}{6}$$

$$(\sqrt{x})' = \frac{1}{2\sqrt{x}}$$

3) $\lim_{x\to 0} \frac{x^n}{e^x} = \frac{0}{1} = 0$ (brez L'H)

$\lim_{x\to\infty} \frac{x^n}{e^x} = \frac{\infty}{\infty} \overset{L'H}{=} \lim_{x\to\infty} \frac{n\cdot x^{n-1}}{e^x} \overset{L'H}{=}$ $\quad n\cdot x^{n-1} \to \infty$ $(n \neq 1)$, $\quad e^x \to \infty$

$= \lim_{x\to\infty} \frac{n\cdot(n-1)x^{n-2}}{e^x} \overset{n\text{-krat}}{=} \lim_{x\to\infty} \frac{n!}{e^x} = 0$ $\quad e^x \to \infty$

$n=2$ $\quad \lim_{x\to\infty} \frac{x^2}{e^x} \overset{L'H}{=} \lim_{x\to\infty} \frac{2x}{e^x} \overset{L'H}{=} \lim_{x\to\infty} \frac{2}{e^x} = 0$

$n=3$ (podobno). $\quad \frac{x^3}{e^x} \overset{L'H}{=} \frac{3x^2}{e^x} \overset{L'H}{=} \frac{6x}{e^x} \overset{L'H}{=} \frac{6}{e^x} = 0$

4) $\lim_{x\downarrow 0} \left(\frac{1}{x} - \frac{1}{\sqrt{x}}\right) = |\infty - \infty| =$ $\quad \frac{1}{x} \to \infty$, $\quad \frac{1}{\sqrt{x}} \to \infty$

$= \lim_{x\downarrow 0} \frac{1-\sqrt{x}}{x} = \left|\frac{1}{0}\right| = \infty$ $\quad 1-\sqrt{x} \to 1$, $\quad x \to 0$

(brez L'H!)